Sparse stabilization and optimal control of the Cucker–Smale Model

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Abstract

This article is mainly based the work [7], and it is dedicated to the 60th anniversary of B. Bonnard, held in Dijon in June 2012.

We focus on a controlled Cucker–Smale model in finite dimension. Such dynamics model self-organization and consensus emergence in a group of agents. We explore how it is possible to control this model in order to enforce or facilitate pattern formation or convergence to consensus. In particular, we are interested in designing control strategies that are componentwise sparse in the sense that they require a small amount of external intervention, and also time sparse in the sense that such strategies are not chattering in time. These sparsity features are desirable in view of practical issues.

We first show how very simple sparse feedback strategies can be designed with the use of a variational principle, in order to steer the system to consensus. These feedbacks are moreover optimal in terms of decay rate of some functional, illustrating the general principle according to which "sparse is better". We then combine these results with local controllability properties to get global controllability results. Finally, we explore the sparsity properties of the optimal control minimizing a combination of the distance from consensus and of a norm of the control.

Keywords: Cucker–Smale model, consensus emergence, ℓ_1-norm minimization, local controllability, sparse stabilization, sparse optimal control.

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1 Introduction

1.1 Self-organization Vs organization via intervention

In recent years there has been a very fast growing interest in defining and analyzing mathematical models of multiple interacting agents in social dynamics. Usually individual based models, described by suitable dynamical systems, constitute the basis for developing continuum descriptions of the agent distribution, governed by suitable partial differential equations. There are many inspiring applications, such as animal behavior, where the coordinated movement of groups, such as birds (starlings, geese,

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etc.), fishes (tuna, capelin, etc.), insects (locusts, ants, bees, termites, etc.) or certain mammals (wildebeasts, sheep, etc.) is considered, see, e.g., [1, 5, 17, 18, 43, 44, 45, 52, 58, 60] or the review chapter [8], and the numerous references therein. Models in microbiology, such as the Patlak-Keller-Segel model [36, 46], describing the chemotactical aggregation of cells and multicellular micro-organisms, inspired a very rich mathematical literature [32, 33, 48], see also the very recent work [3] and references therein. Human motion, including pedestrian and crowd modeling [20, 21, 39, 42], for instance in evacuation process simulations, has been a matter of intensive research, connecting also with new developments such as mean field games, see [37] and the overview in its Section 2. Certain aspects of human social behavior, as in language evolution [22, 24, 35] or even criminal activities [54], are also subject of intensive study by means of dynamical systems and kinetic models. Moreover, relevant results appeared in the economical realm with the theoretical derivation of wealth distributions [26] and, again in connection with game theory, the description of formation of volatility in financial markets [38]. Beside applications where biological agents, animals and micro-(multi)cellular organisms, or humans are involved, also more abstract modeling of interacting automatic units, for instance simple robots, are of high practical interest [11, 34, 56, 40, 47, 53].

One of the leading concepts behind the modeling of multiagent interaction in the past few years has been *self-organization* [5, 43, 44, 45, 58], which, from a mathematical point of view, can be described as the formation of patterns, to which the systems tend naturally to be attracted. The fascinating mechanism to be revealed by such a modeling is how to connect the microscopical and usually binary rules or social forces of interaction between individuals with the eventual global behavior or group pattern, forming as a superposition in time of the different microscopical effects. Hence, one of the interesting issues of such socio-dynamical models is the global convergence to stable patterns or, as more often and more realistically, the instabilities and local convergence [48].

While the description of pattern formation can explain some relevant real-life behaviors, it is also of paramount interest how one may enforce and stabilize pattern formation in those situations where global and stable convergence cannot be ensured, especially in presence of noise [63], or, vice versa, how one can avoid certain rare and dangerous patterns to form, despite that the system may suddenly tend stably to them. The latter situations may refer, for instance, to the so-called "black swans", usually referred to critical (financial or social) events [2, 57]. In all these situations where the independent behavior of the system, despite its natural tendencies, does not realize the desired result, the active intervention of an external policy maker is essential. This naturally raises the question of which optimal policy should be considered.

In information theory, the best possible way of representing data is usually the most economical for reliably or robustly storing and communicating data. One of the modern ways of describing economical description of data is their *sparse* representation with respect to an adapted *dictionary* [41, Chapter 1]. In this paper we shall translate these concepts to realize *best policies* in stabilization and control of dynamical systems modeling multiagent interactions. Beside stabilization strategies in collective behavior already considered in the recent literature, see e.g. [51, 53], the conceptually closest work to our approach is perhaps the seminal paper [40], where externally driven "virtual leaders" are inserted in a collective motion dynamics in order to enforce a certain behavior. Nevertheless our modeling still differs significantly from this mentioned literature, because we allow us directly, externally, and instantaneously to control the individuals of the group, with no need of introducing predetermined virtual leaders, and we shall specifically seek for the most economical (sparsest) way of leading the group to a certain behavior. In particular, we will mathematically model *sparse controls*, designed to promote the minimal amount of intervention of an external policy maker, in order to enforce nevertheless the formation of certain interesting patterns. In other words we shall activate in time the minimal amount of parameters, potentially limited to certain admissible classes, which can provide a certain verifiable outcome of our system. The relationship between parameter choices and result will be usually highly nonlinear, especially for several known dynamical systems, modeling social dynamics. Was this relationship linear instead, then a rather well-established theory predicts how many degrees of freedom are minimally necessary to achieve the expected outcome, and, depending on certain spectral properties of the linear model, allows also for efficient algorithms to compute them. This theory is known in mathematical signal processing under the name of *compressed sensing*, see the seminal work [6] and [25], see also the review chapter [29]. The major contribution of these papers was to realize that one can combine the power of convex optimization, in particular ℓ_1-norm minimization, and spectral properties of random linear models in order to show optimal results on the ability of ℓ_1-norm minimization of recovering robustly sparsest solutions. Borrowing a page from compressed sensing, we will model sparse stabilization and control strategies by penalizing the class of vector valued controls $u = (u_1, \ldots, u_N) \in (\mathbb{R}^d)^N$ by means of a mixed $\ell_1^N - \ell_2^d$-norm, i.e.,

$$\sum_{i=1}^{N} \|u_i\|,$$

where here $\|\cdot\|$ is the ℓ_2^d-Euclidean norm on $\mathbb{R}^d$. This mixed norm has been used for instance in [28] as a *joint sparsity* constraint and it has the effect of optimally sparsifying multivariate vectors in compressed sensing problems [27]. The use of (scalar) ℓ_1-norms to penalize controls dates back to the 60's with the models of linear fuel consumption [19]. More recent work in dynamical systems [61] resumes again ℓ_1-minimization emphasizing its sparsifying power. Also in optimal control with partial differential equation constraints it became rather popular to use L_1-minimization to enforce sparsity of controls [9, 13, 14, 31, 49, 55, 62].

Differently from this previously mentioned work, we will investigate in this paper optimally sparse stabilization and control to enforce pattern formation or, more precisely, convergence to attractors in dynamical systems modeling multiagent interaction. A simple, but still rather interesting and prototypical situation is given by the individual based particle system we are considering here as a particular case

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \frac{v_j - v_i}{(1 + \|x_j - x_i\|^2)^\beta} \end{cases} \tag{1}$$

for $i = 1, \ldots, N$, where $\beta > 0$ and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$ are the *state and consensus parameters* respectively. Here one may want to imagine that the v_i's actually represent abstract quantities such as words of a communication language, opinions, invested capitals, preferences, but also more classical physical quantities such as the velocities in a collective motion dynamics. This model describes the *emerging of consensus* in a group of N interacting agents described by $2d$ degrees of freedom each, trying to align the consensus parameters v_i (also in terms of abstract consensus) with their social neighbors. One of the motivations of this model proposed by Cucker and Smale was in fact to describe the formation and evolution of languages [22, Section 6], although, due to its simplicity, it has been eventually related mainly to the description of the emergence of *consensus* in a group of moving agents, for instance flocking in a swarm of birds [23]. One of the interesting features of this simple system is its rather complete analytical description in terms of its ability of convergence to attractors according to the parameter $\beta > 0$ which is ruling the *communication rate* between far distant agents. For $\beta \leqslant \frac{1}{2}$, corresponding to a still rather strong long - social - distance interaction, for every initial condition the system will converge to a consensus pattern, characterized by the fact that all the parameters $v_i(t)$'s will tend for $t \to +\infty$ to the mean $\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i(t)$ which is actually an invariant of the dynamics. For $\beta > \frac{1}{2}$, the emergence of consensus happens only under certain configurations of state variables and consensus parameters, i.e., when the group is sufficiently close to its state center of mass or when the consensus parameters are sufficiently close to their mean. Nothing instead can be said a priori when at the same time one has $\beta > \frac{1}{2}$ and the mentioned conditions on the initial data are not fulfilled. Actually one can easily construct counterexamples to formation of consensus, see our Example 1 below. In this situation, it is interesting to consider external control strategies which will facilitate the formation of consensus, which is precisely the scope of this work.

1.2 The general Cucker-Smale model and introduction to its control

Let us introduce the general Cucker–Smale model under consideration in this article.

We consider a system of N interacting agents. The state of each agent is described by a pair (x_i, v_i) of vectors of the Euclidean space $\mathbb{R}^d$, where x_i represents the *main state* of an agent and the v_i its *consensus parameter*. The main state of the group of N agents is given by the N-uple $x = (x_1, \ldots, x_N)$. Similarly for the consensus parameters $v = (v_1, \ldots, v_N)$. The space of main states and the space of consensus parameters is $(\mathbb{R}^d)^N$ for both, the product N-times of the Euclidean space $\mathbb{R}^d$ endowed with the induced inner product structure.

The time evolution of the state (x_i, v_i) of the i^{th} agent is governed by the equations

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)), \end{cases} \tag{2}$$

for every $i = 1, \ldots, N$, where $a \in C^1([0, +\infty))$ is a *nonincreasing positive function*. Here, $\|\cdot\|$ denotes again the ℓ_2^d-Euclidean norm in $\mathbb{R}^d$. The meaning of the second equation is that each agent adjusts its consensus parameter with those of other agents in relation with a weighted average of the differences. The influence of the j^{th} agent on the dynamics of the i^{th} is a function of the (social) distance of the two agents. Note that the mean consensus parameter $\bar{v} = \frac{1}{N}\sum_{i=1}^N v_i(t)$ is an invariant of the dynamics, hence it is constant in time.

For every $v \in (\mathbb{R}^d)^N$, we define the mean vector $\bar{v} = \frac{1}{N}\sum_{i=1}^N v_i$ and the symmetric bilinear form B on $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ by

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. We set

$$\mathcal{V}_f = \{(v_1, \ldots, v_N) \in (\mathbb{R}^d)^N \mid v_1 = \cdots = v_N\}, \text{ and } \mathcal{V}_\perp = \{(v_1, \ldots, v_N) \in (\mathbb{R}^d)^N \mid \sum_{i=1}^{N} v_i = 0\}. \tag{3}$$

These are two orthogonal subspaces of $(\mathbb{R}^d)^N$. Every $v \in (\mathbb{R}^d)^N$ can be written as $v = v_f + v_\perp$ with $v_f = (\bar{v}, \ldots, \bar{v}) \in \mathcal{V}_f$ and $v_\perp \in \mathcal{V}_\perp$. Note that B restricted to $\mathcal{V}_\perp \times \mathcal{V}_\perp$ coincides, up to the factor $1/N$, with the scalar product on $(\mathbb{R}^d)^N$.

Given a solution $(x(t), v(t))$ of (2) we define the *dispersion*

$$X(t) := B(x(t), x(t)) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2,$$

and the *disagreement*

$$V(t) := B(v(t), v(t)) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|v(t)_{\perp_i}\|^2.$$

Consensus is a state in which all agents have the same consensus parameter, that is, all agents have reached an agreement.

Definition 1 (Consensus point)**.** *A steady configuration of System* (2) $(x,v) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$ *is called a* consensus point *in the sense that the dynamics originating from* (x,v) *is simply given by rigid translation* $x(t) = x + t\bar{v}$*. We call* $(\mathbb{R}^d)^N \times \mathcal{V}_f$ *the* consensus manifold*.*

The dynamics originating from a consensus point $(x,v) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$ is simply given by a rigid translation of the main state $x(t) = x + t\bar{v}$.

Definition 2 (Consensus)**.** *We say that a solution* $(x(t), v(t))$ *of System* (2) *tends to* consensus *if the consensus parameter vectors tend to the mean* $\bar{v} = \frac{1}{N}\sum_i v_i$*, namely if* $\lim_{t\to\infty} v_i(t) = \bar{v}$ *for every* $i = 1, \ldots, N$*.*

Note that, because of uniqueness, a solution of (2) cannot reach consensus within finite time, unless the initial datum is already a consensus point. The consensus manifold is invariant for (2).

For multi-agent systems of the form (2) sufficient conditions for consensus emergence are a particular case of the main result in [30] and are given in the following proposition.

Proposition 1. *Let* $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *be such that* $X_0 = B(x_0, x_0)$ *and* $V_0 = B(v_0, v_0)$ *satisfy*

$$\int_{\sqrt{X_0}}^{\infty} a(\sqrt{2N}r)dr \geqslant \sqrt{V_0}\,. \tag{4}$$

Then the solution of (2) *with initial data* (x_0, v_0) *tends to consensus.*

In the following we call the subset of $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ satisfying (4) the *consensus region*, which represents the basin of attraction of the consensus manifold. Notice that the condition (4) is actually satisfied as soon as V_0 and X_0 are sufficiently small, i.e., the system has initially sufficient concentration in the consensus parameters and in the main states.

Although consensus forms a rigidly translating stable pattern for the system and represents in some sense a "convenient" choice for the group, there are initial conditions for which the system does not tend to consensus, as the following example shows.

Example 1 (Cucker–Smale system: two agents on the line)**.** *Consider the Cucker–Smale system* (2)*, with* $a(x) = 1/(1 + x^2)$ *in the case of two agents moving on* $\mathbb{R}$ *with position and velocity at time* t*,* $(x_1(t), v_1(t))$ *and* $(x_2(t), v_2(t))$*. Assume for simplicity that* $\beta = 1, K = 2$*, and* $\sigma = 1$*. Let* $x(t) = x_1(t) - x_2(t)$ *be the relative main state and* $v(t) = v_1(t) - v_2(t)$ *be the relative consensus parameter. Equation* (2)*, then reads*

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\dfrac{v}{1+x^2} \end{cases}$$

with initial conditions $x(0) = x_0$ *and* $v(0) = v_0 > 0$*. The solution of this system can be found by direct integration, as from* $\dot{v} = -\dot{x}/(1 + x^2)$ *we have*

$$v(t) - v_0 = -\arctan x(t) + \arctan x_0.$$

If the initial conditions satisfy $\arctan x_0 + v_0 < \pi/2$ *then the relative main state* $x(t)$ *is bounded uniformly by* $\tan(\arctan x_0 + v_0)$*, and the boundedness of the state variables is sufficient for consensus. If* $\arctan x_0 + v_0 = \pi/2$ *then the system tends to consensus as well since* $v(t) = \pi/2 - \arctan x(t)$*.*

On the other hand, whenever $\arctan x_0 + v_0 > \pi/2$*, which implies* $\arctan x_0 + v_0 \geqslant \pi/2 + \varepsilon$ *for some* $\varepsilon > 0$*, the consensus parameter* $v(t)$ *remains bounded away from* 0 *for every time, since*

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0 \geqslant -\arctan x(t) + \pi/2 + \varepsilon > \varepsilon,$$

for every $t > 0$*. In other words, the system does not tend to consensus.*

When the consensus in a group of agents is not achieved by self-organization of the group, as in Example 1, it is natural to ask whether it is possible to induce the group to reach it by means of an external action. In this sense we introduce the notion of *organization via intervention*. We consider the system (2) of N interacting agents, in which the dynamics of every agent are moreover subject to the action of an external field. Admissible controls, accounting for the external field, are measurable functions $u = (u_1, \dots, u_N) : [0, +\infty) \to (\mathbb{R}^d)^N$ satisfying the $\ell_1^N - \ell_2^d$-norm constraint

$$\sum_{i=1}^{N} \|u_i(t)\| \leqslant M, \tag{5}$$

for every $t > 0$, for a given positive constant M. The time evolution of the state is governed by

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t), \end{cases} \tag{6}$$

for $i = 1, \dots, N$, and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$.

Our aim is then to find admissible controls steering the system to the consensus region in finite time. In particular, we shall address the question of quantifying the minimal amount of intervention one external policy maker should use on the system in order to lead it to consensus, and we formulate a practical strategy to approach optimal interventions.

2 Sparse Feedback Control of the Cucker-Smale Model

2.1 A first result of stabilization

Note first that if the integral $\int_0^\infty a(r)dr$ diverges then every pair $(X, V) > 0$ satisfies (4), in other words the interaction between the agents is so strong that the system will reach the consensus no matter what the initial conditions are. In this section we are interested in the case where consensus does not arise autonomously therefore throughout this section we will assume that

$$a \in L^1(0, +\infty).$$

The quantity $V(t)$ is, in fact, a Lyapunov functional for the uncontrolled System (2). For the controlled System (6) such quantity actually depends on the choice of the control, which can be properly optimized. As a first observation we prove that an appropriate choice of the control law can always stabilize the system to consensus. Indeed, it is easy to see that, for every $M > 0$ and for every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, the feedback control defined pointwise in time by $u(t) = -\alpha v_\perp(t)$, with $0 < \alpha \leqslant \frac{M}{N\sqrt{B(v_0,v_0)}}$, satisfies the constraint (5) for every $t \geqslant 0$ and stabilizes the system (6) to consensus (in infinite time). In other words the system (5)-(6) is (semi-globally) feedback stabilizable. Nevertheless this result has a merely theoretical value: the feedback stabilizer $u = -\alpha v_\perp$ is not convenient for practical purposes since it requires to act at every instant of time on all the agents in order to steer the system to consensus, which may require a large amount of instantaneous communications. In what follows we address the design of more economical and practical feedback controls which can be both componentwise and time sparse.

2.2 Componentwise sparse feedback stabilization

We introduce here a variational principle leading to a componentwise sparse stabilizing feedback law.

Definition 3. *For every $M > 0$ and every $(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, let $U(x, v)$ be defined as the set of solutions of the variational problem*

$$\min \left(B(v,u) + \gamma(B(x,x)) \frac{1}{N} \sum_{i=1}^{N} \|u_i\| \right) \qquad \textit{subject to } \sum_{i=1}^{N} \|u_i\| \leqslant M\,, \tag{7}$$

where

$$\gamma(X) = \int_{\sqrt{X}}^{\infty} a(\sqrt{2N} r) dr. \tag{8}$$

The meaning of (7) is the following. Minimizing the component $B(v, u) = B(v_\perp, u)$ means that, at every instant of time, the control $u \in U(x, v)$ is of the form $u = -\alpha \cdot v_\perp$, for some $\alpha = (\alpha_1, \dots, \alpha_N)$ sequence of nonnegative scalars. Hence it acts as an additional force which pulls the particles towards having the same mean consensus parameter. Imposing additional $\ell_1^N - \ell_2^d$-norm constraints has the function of enforcing *sparsity*, i.e., most of the $\alpha_i's$ will turn out to be zero, as we will in more detail clarify below. Eventually, the threshold $\gamma(X)$ is chosen in such a way that when the control switches-off the criterion (4) is fulfilled.

The componentwise sparsity feature of feedback controls $u(x, v) \in U(x, v)$ is analyzed in the next remark, where we make explicit the set $U(x, v)$ according to the value of (x, v) in a partition of the space $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$.

Remark 1. First of all, it is easy to see that, for every $(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and every element $u(x, v) \in U(x, v)$ there exist nonnegative real numbers α_i's such that

$$u_i(x,v) = \begin{cases} 0 & \text{if } v_{\perp_i} = 0, \\ -\alpha_i \dfrac{v_{\perp_i}}{\|v_{\perp_i}\|} & \text{if } v_{\perp_i} \neq 0, \end{cases} \tag{9}$$

where $0 \leqslant \sum_{i=1}^N \alpha_i \leqslant M$.

The componentwise sparsity of u depends on the possible values that the α_i's may take in function of (x, v). Actually, the space $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ can be partitioned in the union of the four disjoint subsets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, and $\mathcal{C}_4$ defined by

$\mathcal{C}_1 = \{(x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leqslant i \leqslant N} \|v_{\perp_i}\| < \gamma(B(x,x))\}$,

$\mathcal{C}_2 = \{(x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leqslant i \leqslant N} \|v_{\perp_i}\| = \gamma(B(x,x))\}$,

$\mathcal{C}_3 = \{(x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leqslant i \leqslant N} \|v_{\perp_i}\| > \gamma(B(x,x))$ and there exists a unique $i \in \{1, \dots, N\}$ such that $\|v_{\perp_i}\| > \|v_{\perp_j}\|$ for every $j \neq i\}$,

$\mathcal{C}_4 = \{(x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mid \max_{1 \leqslant i \leqslant N} \|v_{\perp_i}\| > \gamma(B(x,x))$ and there exist $k \geqslant 2$ and $i_1, \dots, i_k \in \{1, \dots, N\}$ such that $\|v_{\perp_{i_1}}\| = \cdots = \|v_{\perp_{i_k}}\|$ and $\|v_{\perp_{i_1}}\| > \|v_{\perp_j}\|$ for every $j \notin \{i_1, \dots, i_k\}\}$.

The subsets $\mathcal{C}_1$ and $\mathcal{C}_3$ are open, and the complement $(\mathcal{C}_1 \cup \mathcal{C}_3)^c$ has Lebesgue measure zero. Moreover for every $(x, v) \in \mathcal{C}_1 \cup \mathcal{C}_3$, the set $U(x, v)$ is single valued and its value is a sparse vector with at most one nonzero component. More precisely, one has $U|_{\mathcal{C}_1} = \{0\}$ and $U|_{\mathcal{C}_3} = \{(0, \dots, 0, -M v_{\perp_i}/\|v_{\perp_i}\|, 0, \dots, 0)\}$ for some unique $i \in \{1, \dots, N\}$.

If $(x, v) \in \mathcal{C}_2 \cup \mathcal{C}_4$ then a control in $U(x, v)$ may not be sparse: indeed in these cases the set $U(x, v)$ consists of all $u = (u_1, \dots, u_N) \in (\mathbb{R}^d)^N$ such that $u_i = -\alpha_i v_{\perp_i}/\|v_{\perp_i}\|$ for every $i = 1, \dots, N$, where the α_i's are nonnegative real numbers such that $0 \leqslant \sum_{i=1}^N \alpha_i \leqslant M$ whenever $(x, v) \in \mathcal{C}_2$, and $\sum_{i=1}^N \alpha_i = M$ whenever $(x, v) \in \mathcal{C}_4$.

By showing that the choice of feedback controls as in Definition 3 optimizes the Lyapunov functional $V(t)$, we can again prove convergence to consensus.

Theorem 1 ([7]). *For every $(x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, and $M > 0$, set $F(x,v) = \{(v, -L_x v + u) \mid u \in U(x,v)\}$, where $U(x,v)$ is as in Definition 3. Then for every initial pair $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, the differential inclusion*

$$(\dot{x}, \dot{v}) \in F(x,v) \tag{10}$$

with initial condition $(x(0), v(0)) = (x_0, v_0)$ is well-posed and its solutions converge to consensus as t tends to $+\infty$.

Remark 2. By definition of the feedback controls $u(x,v) \in U(x,v)$, and from Remark 1, it follows that, along a closed-loop trajectory, as soon as $V(t)$ is small enough with respect to $\gamma(B(x,x))$ the trajectory has entered the consensus region defined by (4). From this point in time no action is further needed to stabilize the system, since Proposition 1 ensures then that the system is naturally stable to consensus. When the system enters the region $\mathcal{C}_1$ the control switches-off automatically by being set to 0 forever. In [7] it is proved that the time T needed to steer the system to the consensus region is not larger than $\frac{N}{M}\left(\sqrt{V(0)} - \gamma(\bar{X})\right)$, where γ is defined by (8), and $\bar{X} = 2X(0) + \frac{N^4}{2M^2}V(0)^2$.

Within the set $U(x,v)$ as in Definition 3, which in general does not contain only sparse solutions, there are actually selections with maximal sparsity.

Definition 4. *We select the* componentwise sparse feedback control $u^\circ = u^\circ(x,v) \in U(x,v)$ *according to the following criterion:*

- *if* $\max_{1 \leqslant i \leqslant N} \|v_{\perp_i}\| \leqslant \gamma(B(x,x))^2$, *then* $u^\circ = 0$,
- *if* $\max_{1 \leqslant i \leqslant N} \|v_{\perp_i}\| > \gamma(B(x,x))^2$ *let* $j \in \{1, \ldots, N\}$ *be the smallest index such that*

$$\|v_{\perp_j}\| = \max_{1 \leqslant i \leqslant N} \|v_{\perp_i}\|$$

  then

$$u_j^\circ = -M \frac{v_{\perp_j}}{\|v_{\perp_j}\|}, \qquad \textit{and} \qquad u_i^\circ = 0 \quad \textit{for every } i \neq j.$$

The control u° can be, in general, highly irregular. If we consider for instance a system in which there are two agents with maximal disagreement then the control u° switches at every instant from one agent to the other and it is everywhere discontinuous. The natural definition of solution associated with the feedback control u° is therefore the notion of sampling solution as introduced in [12].

Definition 5 (Sampling solution). *Let $U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ be continuous and locally Lipschitz in x uniformly on compact subset of $\mathbb{R}^n \times U$. Given a feedback $u : \mathbb{R}^n \to U$, $\tau > 0$, and $x_0 \in \mathbb{R}^n$ we define the* sampling solution *of the differential system*

$$\dot{x} = f(x, u(x)), \quad x(0) = x_0,$$

as the continuous (actually piecewise C^1) function $x : [0,T] \to \mathbb{R}^n$ solving recursively for $k \geqslant 0$

$$\dot{x}(t) = f(x(t), u(x(k\tau))), \quad t \in [k\tau, (k+1)\tau]$$

using as initial value $x(k\tau)$, the endpoint of the solution on the preceding interval, and starting with $x(0) = x_0$. We call τ the sampling time.

This notion of solution is of particular interest for applications in which a minimal interval of time between two switchings of the control law is demanded. As the sampling time becomes smaller and smaller the sampling solution of (6) associated with our componentwise sparse control u° as defined in Definition 4 approaches uniformly a Filippov solution of (10), i.e. an absolutely continuous function satisfying (10) for almost every t. In particular we have the following statement.

Theorem 2 ([7]). *Let u° be the componentwise sparse control defined in Definition 4. For every $M > 0, \tau > 0$, and $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ let $(x_\tau(t), v_\tau(t))$ be the sampling solution of* (6) *associated with u°. Then $(x_\tau(t), v_\tau(t))$ tends uniformly to a Filippov solution of* (10) *as τ tends to* 0.

Let us stress that, as a byproduct of our analysis, we shall eventually construct practical feedback controls which are both *componentwise* and *time sparse*.

2.3 Time sparse feedback stabilization

Theorem 1 gives the existence of a feedback control whose behavior may be, in principle, very complicated and that may be nonsparse. In this section we are going to exploit the variational principle (7) to give an explicit construction of a piecewise constant and componentwise sparse control steering the system to consensus. The idea is to take a selection of a feedback in $U(x, v)$ which has *at most one nonzero component* for every $(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, as in Definition 4, and then sample it to avoid *chattering* phenomena (see, e.g., [64]).

Theorem 3 ([7]). *Fix $M > 0$ and consider the control u° law given by Definition 4. Then for every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ there exists $\tau_0 > 0$ small enough, such that for all $\tau \in (0, \tau_0]$ the sampling solution of* (6) *associated with the control u°, the sampling time τ, and initial pair (x_0, v_0) reaches the consensus region in finite time.*

Remark 3. The maximal sampling time τ_0 depends on the number of agents N, the $\ell_1^N - \ell_2^d$-norm bound M on the control, the initial conditions (x_0, v_0), and the rate of communication function $a(\cdot)$. In [7] the explicit bound on τ_0 is given by

$$\tau_0 \left(a(0)(1 + \sqrt{N})\sqrt{B(v_0, v_0)} + M \right) + \tau_0^2 2a(0)M \leqslant \frac{\gamma(\bar{X})}{2}, \tag{11}$$

where $\bar{X} = 2B(x_0, x_0) + \frac{2N^4}{M^2} B(v_0, v_0)^2$. Moreover, as in Remark 2, the sampled control is switched-off as soon as the sampled trajectory enters the region $\mathcal{C}_1$. In particular the systems reaches the consensus region defined by (4) within time $T \leqslant T_0 = \frac{2N}{M}(\sqrt{V(0)} - \gamma(\bar{X}))$. The control is then set to zero in a time that is not larger than $\frac{2\sqrt{N}}{M}(\sqrt{N}\sqrt{V(0)} - \gamma(\bar{X}))$.

3 Sparse is Better

3.1 Instantaneous optimality of componentwise sparse controls

The componentwise sparse control u° of Definition 4 corresponds to the strategy of acting, at each instant of time, on the agent whose consensus parameter is farthest from the mean and to steer it to consensus. Since this control strategy is designed to act on at most one agent at each time, we claim that in some sense it is instantaneously the "best one". To clarify this notion of *instantaneous optimality* which also implies its *greedy* nature, we shall compare this strategy with all other feedback strategies $u(x, v) \in U(x, v)$ and discuss their efficiency in terms of the instantaneous decay rate of the functional V.

Proposition 2 ([7]). *The feedback control $u^\circ(t) = u^\circ(x(t), v(t))$ of Definition 4, associated with the solution $((x(t), v(t))$ of Theorem 2, is a minimizer of*

$$\mathscr{R}(t,u) = \frac{d}{dt} V(t),$$

over all possible feedback controls in $U(x(t), v(t))$. In other words, the feedback control u° is the best choice in terms of the rate of convergence to consensus.

This result is somewhat surprising with respect to the perhaps more intuitive strategy of activating controls on more agents. This can be viewed as a mathematical description of the general principle:

> *A policy maker, who is not allowed to have prediction on future developments, should always consider more favorable to intervene with stronger actions on the fewest possible instantaneous optimal leaders than trying to control more agents with minor strength.*

4 Sparse Controllability Near the Consensus Manifold

In this section we address the problem of controllability near the consensus manifold. The stabilization results of Section 2 provide a constructive strategy to stabilize the multi-agent system (6): the system is first steered to the region of consensus, and then in free evolution reaches consensus in infinite time. Here we study the local controllability near consensus, and infer a global controllability result to consensus.

The following result states that, almost everywhere, local controllability near the consensus manifold is possible by acting on only one arbitrary component of a control, in other words whatever is the controlled agent it is possible to steer a group, sufficiently close to a consensus point, to any other point sufficiently close. The main ingredient is the Kalman controllability condition for the linearized system. Recall that the consensus manifold is $(\mathbb{R}^d)^N \times \mathcal{V}_f$, where $\mathcal{V}_f$ is defined by (3).

Proposition 3 ([7]). *For every $M > 0$, for almost every $\tilde{x} \in (\mathbb{R}^d)^N$ and for every $\tilde{v} \in \mathcal{V}_f$, for every time $T > 0$, there exists a neighborhood W of $(\tilde{x}, \tilde{v})$ in $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ such that, for all points (x_0, v_0) and (x_1, v_1) of W, for every index $i \in \{1, \dots, N\}$, there exists a componentwise and time sparse control u satisfying the constraint (5), every component of which is zero except the i^{th} (that is, $u_j(t) = 0$ for every $j \neq i$ and every $t \in [0,T]$), steering the control system (6) from (x_0, v_0) to (x_1, v_1) in time T.*

As a consequence of this local controllability result, we infer that we can steer the system from any consensus point to almost any other one by acting only on one agent. This is a partial but global controllability result, whose proof follows the strategy developed in [15, 16] for controlling heat and wave equations on steady-states.

Now, it follows from the results of the previous section that we can steer any initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ to the consensus region defined by (4), by means of a componentwise and time sparse control. Once the trajectory has entered this region, the system converges naturally (i.e., without any action: $u = 0$) to some point of the consensus manifold $(\mathbb{R}^d)^N \times \mathcal{V}_f$, in infinite time. This means that, for some time large enough, the trajectory enters the neighborhood of controllability whose existence is claimed in Proposition 3, and hence can be steered to the consensus manifold within finite time. Since the consensus manifold is connected then there exists a control able move the system in order to reach almost any other desired consensus point. Hence we have obtained the following.

Corollary 1 ([7]). *For every $M > 0$, for every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for almost every $(x_1, v_1) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$, there exist $T > 0$ and a componentwise and time sparse control $u : [0,T] \to (\mathbb{R}^d)^N$, satisfying (5), such that the corresponding solution starting at (x_0, v_0) arrives at the consensus point (x_1, v_1) within time T.*

5 Sparse Optimal Control of the Cucker-Smale Model

In this section we investigate the sparsity properties of a *finite time optimal control* with respect to a cost functional involving the discrepancy of the state variables to consensus and a $\ell_1^N - \ell_2^d$-norm term of the control.

While the *greedy strategies* based on instantaneous feedback as presented in Section 2 models the more realistic situation where the policy maker is not allowed to make future predictions, the optimal control problem presented in this section actually describes a model where the policy maker is allowed to see how the dynamics can develop. Although the results of this section do not lead systematically to sparsity, what is interesting to note is that the *sparsity* of the optimal control is actually encoded in terms of the codimension of certain manifolds, which have actually null Lebesgue measure in the space of cotangent vectors.

We consider the optimal control problem of determining a trajectory solution of (6), starting at $(x(0), v(0)) = (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, and minimizing a cost functional which is a combination of the distance from consensus with the $\ell_1^N - \ell_2^d$-norm of the control (as in [27, 28]), under the control constraint (5). More precisely, the cost functional considered here is, for a given $\gamma > 0$,

$$\int_0^T \sum_{i=1}^N \left(\Big(v_i(t) - \frac{1}{N} \sum_{j=1}^N v_j(t) \Big)^2 + \gamma \sum_{i=1}^N \|u_i(t)\| \right) dt. \tag{12}$$

Using classical results in optimal control theory (see for instance [4, Theorem 5.2.1] or [10, 59]), this optimal control problem has a unique optimal solution $(x(\cdot), v(\cdot))$, associated with a control u on $[0, T]$, which is characterized as follows. According to the Pontryagin Minimum Principle (see [50]), there exist absolutely continuous functions $p_x(\cdot)$ and $p_v(\cdot)$ (called adjoint vectors), defined on $[0, T]$ and taking their values in $(\mathbb{R}^d)^N$, satisfying the adjoint equations

$$\begin{cases} \dot{p}_{x_i} = \dfrac{1}{N} \displaystyle\sum_{j=1}^N \frac{a(\|x_j - x_i\|)}{\|x_j - x_i\|} \langle x_j - x_i, v_j - v_i \rangle (p_{v_j} - p_{v_i}), \\ \dot{p}_{v_i} = -p_{x_i} - \dfrac{1}{N} \displaystyle\sum_{j \neq i} a(\|x_j - x_i\|)(p_{v_j} - p_{v_i}) - 2v_i + \frac{2}{N} \sum_{j=1}^N v_j, \end{cases} \tag{13}$$

almost everywhere on $[0, T]$, and $p_{x_i}(T) = p_{v_i}(T) = 0$, for every $i = 1, \ldots, N$. Moreover, for almost every $t \in [0, T]$ the optimal control $u(t)$ must minimize the quantity

$$\sum_{i=1}^N \langle p_{v_i}(t), w_i \rangle + \gamma \sum_{i=1}^N \|w_i\|, \tag{14}$$

over all possible $w = (w_1, \ldots, w_N) \in (\mathbb{R}^d)^N$ satisfying $\sum_{i=1}^N \|w_i\| \leqslant M$.

In analogy with the analysis in Section 2 we identify five regions $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5$ covering the (cotangent) space $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$:

$\mathcal{O}_1 = \{(x, v, p_x, p_v) \mid \|p_{v_i}\| < \gamma$ for every $i \in \{1, \ldots, N\}\}$,

$\mathcal{O}_2 = \{(x, v, p_x, p_v) \mid$ there exists a unique $i \in \{1, \ldots, N\}$ such that $\|p_{v_i}\| = \gamma$ and $\|p_{v_j}\| < \gamma$ for every $j \neq i\}$,

$\mathcal{O}_3 = \{(x, v, p_x, p_v) \mid$ there exists a unique $i \in \{1, \ldots, N\}$ such that $\|p_{v_i}\| > \gamma$ and $\|p_{v_i}\| > \|p_{v_j}\|$ for every $j \neq i\}$,

$\mathcal{O}_4 = \{(x,v,p_x,p_v) \mid$ there exist $k \geqslant 2$ and $i_1,\dots,i_k \in \{1,\dots,N\}$ such that $\|p_{v_{i_1}}\| = \|p_{v_{i_2}}\| = \cdots = \|p_{v_{i_k}}\| > \gamma$ and $\|p_{v_{i_1}}\| > \|p_{v_j}\|$ for every $j \notin \{i_1,\dots,i_k\}\}$,

$\mathcal{O}_5 = \{(x,v,p_x,p_v) \mid$ there exist $k \geqslant 2$ and $i_1,\dots,i_k \in \{1,\dots,N\}$ such that $\|p_{v_{i_1}}\| = \|p_{v_{i_2}}\| = \cdots = \|p_{v_{i_k}}\| = \gamma$ and $\|p_{v_j}\| < \gamma$ for every $j \notin \{i_1,\dots,i_k\}\}$.

The subsets $\mathcal{O}_1$ and $\mathcal{O}_3$ are open, the submanifold $\mathcal{O}_2$ is closed (and of zero Lebesgue measure) and $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$ is of full Lebesgue measure in $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$. Moreover if an extremal $(x(\cdot),v(\cdot),p_x(\cdot),p_v(\cdot))$ solution of (6)-(13) is in $\mathcal{O}_1 \cup \mathcal{O}_3$ along an open interval of time then the control is uniquely determined from (14) and is componentwise sparse. Indeed, if there exists an interval $I \subset [0,T]$ such that $(x(t),v(t),p_x(t),p_v(t)) \in \mathcal{O}_1$ for every $t \in I$, then (14) yields $u(t) = 0$ for almost every $t \in I$. If $(x(t),v(t),p_x(t),p_v(t)) \in \mathcal{O}_3$ for every $t \in I$ then (14) yields $u_j(t) = 0$ for every $j \neq i$ and $u_i(t) = -M\frac{p_{v_i}(t)}{\|p_{v_i}(t)\|}$ for almost every $t \in I$. Finally, if $(x(t),v(t),p_x(t),p_v(t)) \in \mathcal{O}_2$ for every $t \in I$, then (14) does not determine $u(t)$ in a unique way: it yields that $u_j(t) = 0$ for every $j \neq i$ and $u_i(t) = -\alpha\frac{p_{v_i}(t)}{\|p_{v_i}(t)\|}$ with $0 \leqslant \alpha \leqslant M$, for almost every $t \in I$. However u is still componentwise sparse on I.
The submanifolds $\mathcal{O}_4$ and $\mathcal{O}_5$ are of zero Lebesgue measure. When the extremal is in these regions, the control is not uniquely determined from (14) and is not necessarily componentwise sparse. More precisely, if $(x(t),v(t),p_x(t),p_v(t)) \in \mathcal{O}_4 \cup \mathcal{O}_5$ for every $t \in I$, then (14) is satisfied by every control of the form $u_{i_j}(t) = -\alpha_j\frac{p_{v_{i_j}}(t)}{\|p_{v_{i_j}}(t)\|}$, $j = 1,\dots,k$, and $u_l = 0$ for every $l \notin \{i_1,\dots,i_k\}$, where the α_i's are nonnegative real numbers such that $0 \leqslant \sum_{j=1}^k \alpha_j \leqslant M$ whenever $(x(t),v(t),p_x(t),p_v(t)) \in \mathcal{O}_5$, and such that $\sum_{j=1}^k \alpha_j = M$ whenever $(x(t),v(t),p_x(t),p_v(t)) \in \mathcal{O}_4$.

In [7, Proposition 5], it is proved that the submanifolds $\mathcal{O}_4$ and $\mathcal{O}_5$ are stratified submanifolds (in the sense of Whitney) of codimension larger than or equal to two. More precisely, $\mathcal{O}_4$ (respectively, $\mathcal{O}_5$) is the union of submanifolds of codimension $2(k-1)$ (respectively, $2k$), where k is the index appearing in the definition of these subsets and it is as well the number of active components of the control at the same time.

Therefore, it is interesting to see that the componentwise sparsity features of the optimal control are coded in terms of the codimension of the above submanifolds. Note that, since $p_x(T) = p_v(T) = 0$, there exists $\varepsilon > 0$ such that $u(t) = 0$ for every $t \in [T-\varepsilon, T]$. In other words, at the end of the interval of time the extremal $(x(\cdot),v(\cdot),p_x(\cdot),p_v(\cdot))$ is in $\mathcal{O}_1$.

It is an open question of knowing whether the extremal may lie on the submanifolds $\mathcal{O}_4$ or $\mathcal{O}_5$ along a nontrivial interval of time. What can be obviously said is that, for generic initial conditions $((x_0,v_0),(p_x(0),p_v(0)))$, the optimal extremal does not stay in $\mathcal{O}_4 \cup \mathcal{O}_5$ along an open interval of time; such a statement is however unmeaningful since the pair $(p_x(0),p_v(0))$ of initial adjoint vectors is not arbitrary and is determined through the shooting method by the final conditions $p_x(T) = p_v(T) = 0$.

6 Numerical simulations

In this section we present numerical simulations to study the estimates on the time of action of the feedback stabilizer and on the maximal sampling time given in Remark 3. In general, it is of paramount interest for application to have precise estimates on the sampling time or on the controllability time, in order to study the feasibility of the control processes. A smaller sampling time provides a more precise control law, which, in principle, can steer the system to the consensus region in a smaller time. On the other hand, the smaller the sampling time is the higher the complexity of the control will be.

Throughout the section we consider a system of N agents on $\mathbb{R}^2$ with interaction function $a(x) = 1/(1+x^2)$ and bound on the control $M = 1$. Moreover we consider initial main states all equal to

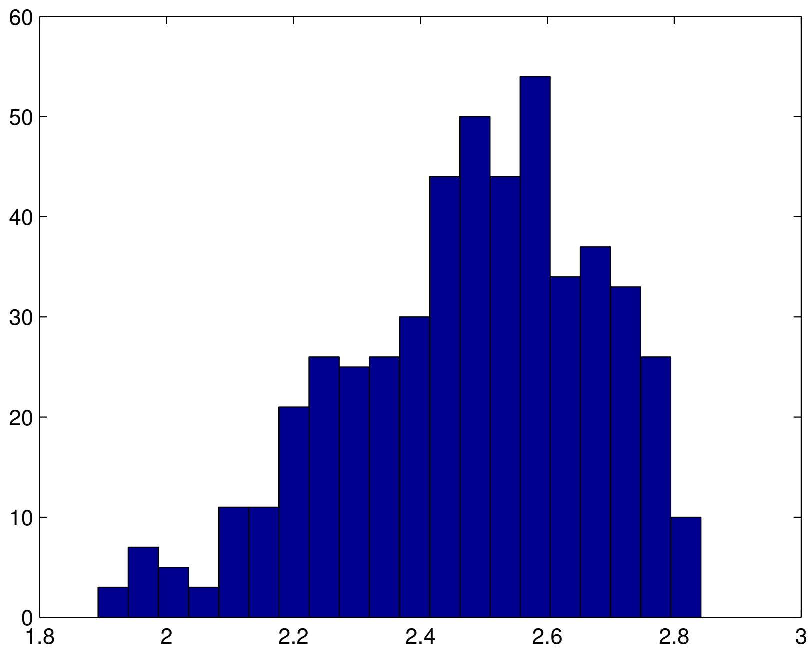

Figure 1: Distribution of the final time of controllability for $N = 5$ and for initial consensus parameters randomly generated 500 times.

the origin. The initial control parameters $v(0)$ are chosen randomly and rescaled in such a way that the initial disagreement is $25\pi^2/(2N)$ that is 100 times the threshold for $B(v(0), v(0))$ given by the sufficient condition for consensus (4) which in this case reads

$$\sqrt{V(0)} \leqslant \int_0^{+\infty} a(\sqrt{2N}r)dr = \frac{\pi}{2\sqrt{2N}}.$$

First, let un consider a system of $N = 5$ agents. In this setting the estimate for the sampling time (11) guarantees that for a sampling time not greater than 1/60 the control scheme converges. Nevertheless, numerically one finds that the system is steered to the consensus region even for quite large sampling times, as, for instance 1/10. We generate randomly the initial condition for the consensus parameters 500 times. The final time of action of the control, for which the system is in the consensus region is always larger than 1.892 and smaller 2.842. The distribution of the final times are represented in the histogram of Figure 1. Note that the estimates of Remark 3 are very far from sharp. Indeed, according to Remark 3, the system enters the consensus in a time not larger than 13.64 and the control switches off in time not larger than 14.78.

Similarly we study the same test with $N = 20$ agents. The sampling time should be smaller than 1/105 according to (11). According to Remark 3 the system enters the consensus region within time 27.2 and the control switches off within time 30.5. We run 250 trials with sampling time 1/50, to improve the computation time. The minimal time of controllability is 1.97 while the maximal is 2.17. Data are collected in Figure 2.

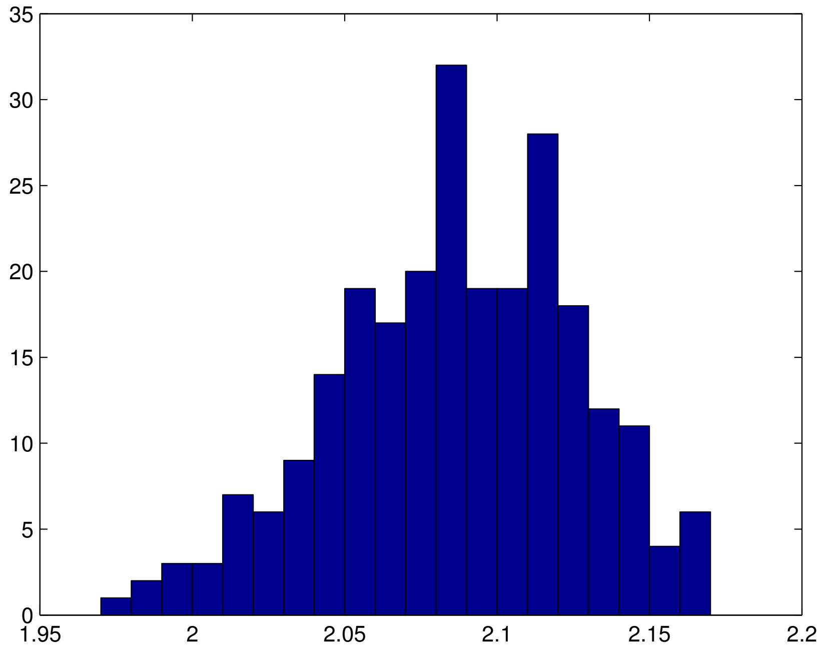

Figure 2: Distribution of the final time of controllability for $N = 20$ and for initial consensus parameters randomly generated 250 times.

Finally, to study the relation between the sampling time and the final time we present some simulation with $N = 20$ agents. The final time depends on the initial conditions and on the geometric structure of the "flock". It is, therefore, in principle very hard to compute a priori the final time. However, it is possible to give some estimates, as in Remark 3. The simulation presented below goes a little further and shows the relation between the sampling time and the final time. The initial consensus parameters are represented in Figure 3, as above the initial disagreement is $25\pi^2/(2N)$. In Figure 4 we present the final time of action of the control as a function of the sampling time. We run 198 simulations for sampling times ranging from 0.006 to 0.2 with an increment of 0.002. We observe that the smallest final time 2.019 is associated with the smallest sampling time 0.006. However, even for bigger sampling time, we have small final times, for instance we have that for a sampling time of 0.04 the final time is 2.02 and for a sampling time of 0.086 the final time is 2.021. We note that there is an "optimal" final time, associated with the Filippov solution as in Theorem 2, which is a lower bound for the controllability time for sampling solutions. On the other hand, in general, the upper bound on the final time increases as the sampling time increases. Studying these two bounds is a very interesting problem in view of the applications.

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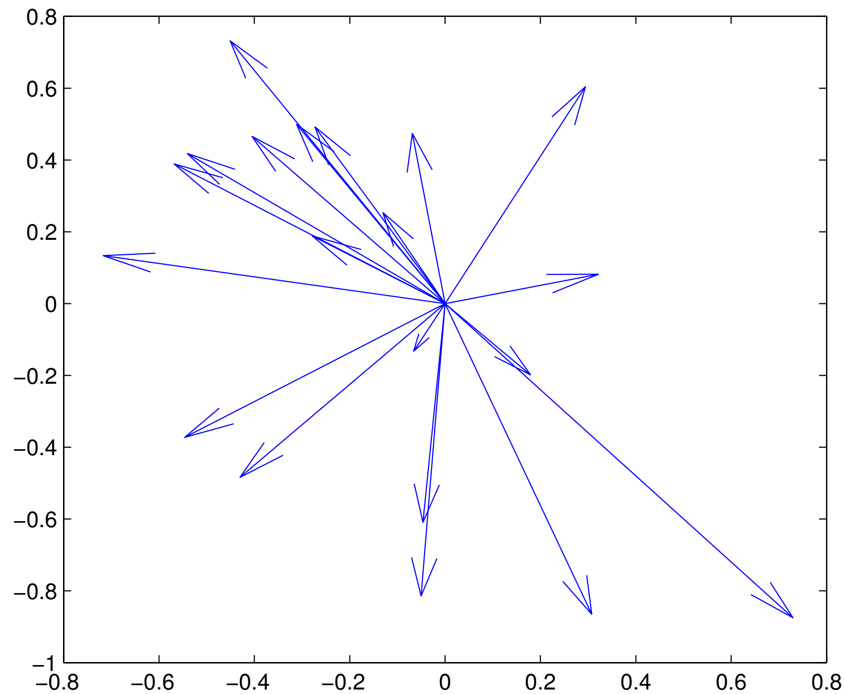

Figure 3: Initial condition for Figure 4

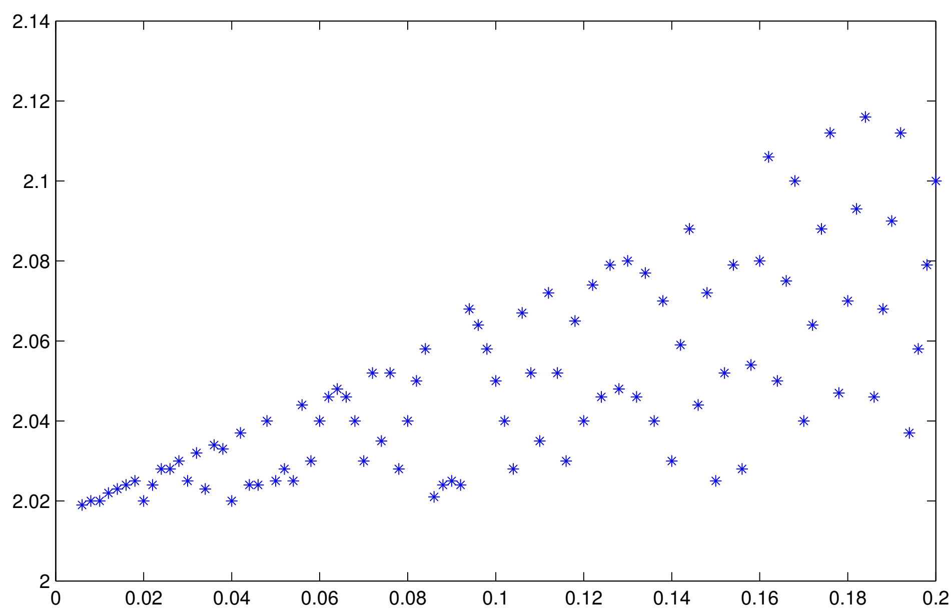

Figure 4: Controllability time in function of the sampling time

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