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Controllability on the Group of Diffeomorphisms

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To my Promise

Contents

Introduction

In this thesis we study sufficient conditions for a family of flows on a smooth manifold M to generate the group $\mathrm{Diff}_0(M)$ of all diffeomorphisms of M that are isotopic to the identity. The problem arises in the framework of control theory. Indeed, consider a driftless control–affine system

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in M \,. \tag{1}$$

Given a family of vector fields $\mathcal{F} = \{f_1, \ldots, f_m\}$, a natural question is to study what kind of dynamics we can realize by an appropriate choice of the time–dependent feedback controls $(u_1(t,q), \ldots, u_m(t,q))$. In particular we will focus on dicrete–time dynamics and, in fact, the problem we treat is to find, given a diffeomorphism P, controls such that the flow of system (1) at a fixed time is, at least approximately, equal to P. This problem is an application of a slightly more general geometrical argument, that is: consider the exponential map $\exp : f \in \mathrm{Vec}\, M \mapsto \exp(f) \in \mathrm{Diff}_0(M)$ mapping a vector field to the flow, at time 1, generated by the equation

$$\dot{q} = f(q), \quad q \in M \,.$$

Under what conditions on a family $\mathcal{F} \subset \mathrm{Vec}\, M$ the group generated by $\exp(\mathcal{F})$ is the whole group $\mathrm{Diff}_0(M)$?
If the manifold M is compact and $\mathcal{F} = \mathrm{Vec}\, M$ then the result follows from the simplicity of the group $\mathrm{Diff}_0(M)$ showed by Thurston in [40]. Indeed, flows are just one-parametric subgroups of $\mathrm{Diff}_0(M)$ and all one-parametric subgroups generate a normal subgroup. In other words, any diffeomorphism of M isotopic to the identity can be presented as composition of exponentials of smooth vector fields.
In the framework of control theory, in the interesting cases, the system cannot evolve along all the possible directions but only along a prescribed vector distribution. If the manifold is connected and the distribution is completely nonholonomic, or *bracket generating* ,then any two points of the manifold can be connected by a curve whose velocity belongs to the distribution. In other words the corresponding control system is *completely controllable.* This is the statement of a classical result in control theory due to Rashevsky [30] and Chow [11].

The main result of this thesis, Theorem 3.1, states that bracket generating distributions provide not only controllability on M but also exact controllability on the group of diffeomorphisms on M. More precisely, any diffeomorphism isotopic to the identity can be presented as a composition of exponentials of vector fields belonging to the prescribed vector distribution. In fact, a stronger result is valid since any diffeomorphism sufficiently close to the identity can be presented as a composition of μ exponentials, where the number μ depends only on the distribution. The exact statement is as follows.

Theorem. *Let M be a compact connected manifold, $\mathcal{F} \subset \operatorname{Vec} M$ be a family of smooth vector fields, and let* $\mathrm{Gr}\mathcal{F} = \{e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} : t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\}$. *If* $\mathrm{Gr}\mathcal{F}$ *acts transitively on M, then there exist a neighborhood $\mathcal{O}$ of the identity in* $\mathrm{Diff}_0(M)$ *and a positive integer μ such that every $P \in \mathcal{O}$ can be presented in the form*

$$P = e^{a_1 f_1} \circ \cdots \circ e^{a_\mu f_\mu} ,$$

for some $f_1, \ldots, f_\mu \in \mathcal{F}$ and $a_1, \ldots, a_\mu \in C^\infty(M)$.

The main strategy to handle this kind of problems consists in studying analytical properties of the exponential map. For example, the result above is a consequence of the fact that a map which is a particular product of exponentials is locally onto. Moreover, natural generalizations arising in control theory lead to investigate how small perturbations of this product of exponentials affect its invertibility. This is the case when studying system (1) dealing with controls of a certain regularity or when considering the system with a drift.

The structure of the thesis is as follows.

In Chapter 1 we introduce the language of geometric control theory and we state some classical results such as the mentioned Rashevsky–Chow Theorem and the Orbit Theorem. In Section 1.9 we prove a first partial result showing that the group generated by the exponentials of vector fields in a bracket generating distribution is dense in the connected component of the identity of the group of diffeomorphisms.

In Chapter 2 we deal with a bracket generating control–affine system with drift f_0 on the real space $\mathbb{R}^d$, i.e.

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d . \tag{2}$$

When studying dynamics of system (2) it is natural to work with time–varying feedback controls. Indeed, if u_i are continuous feedback controls not depending on time then we cannot expect system (2) to have locally asymptotically stable equilibria neither in the case $f_0 = 0$, as it was observed by R. Brockett in [8]. J.-M. Coron suggested to use time–varying feedback controls, periodic with respect to time, for system (2) and proved that asymptotic stability can be successfully achieved by a smooth time–varying feedback (see [12, 13] or [14, section 11.2]). Therefore, since similar results hold true also in the discrete-time case, we need, in order to realize discrete-time dynamics, to work with time–varying feedback controls, even though we do not deal with continuous–time dynamics and stabilization. The main result of this chapter, Theorem 2.6, states that almost every kind of discrete-time dynamics can be realized by an appropriate choice of the time–varying feedback control. More precisely, let $F_{\mathbf{u}} : q(0) \mapsto q(1)$ be the transformation of $\mathbb{R}^d$ which sends the initial value of any solution of system (2) to its value at $t = 1$. Let

$P \in \mathrm{Diff}_0(\mathbb{R}^d)$, $\mathcal{O}_P$ be a C^∞-neighborhood of P, and N be a positive integer, then there exists a time-varying feedback control $\mathbf{u} = (u_1, \ldots, u_m)$ such that $F_{\mathbf{u}} \in \mathcal{O}_P$ and the N-jets of $F_{\mathbf{u}}$ and P at the origin coincide. Therefore the diffeomorphism P can be approximated, in a very strong sense, by a diffeomorphism included in the flow generated by a time–varying feedback control. Moreover, the controls can be taken polynomial with respect to q and trigonometric polynomial with respect to t. The proof makes use of the classical implicit function theorem applied to the map

$$(u_1, \ldots, u_m) \in \mathbf{U} \mapsto J_0^N \left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(t, \cdot) f_i \, dt \right) \in J_0^N(\mathrm{Diff}_0(\mathbb{R}^d)) \,, \tag{3}$$

that, together with the mentioned relaxation result of Section 1.9, guarantees global surjectivity. Finally, by Brouwer fixed point Theorem, it is possible to prove that small perturbations of the map are surjective too. This argument implies that the regularity assumptions for the controls are not restrictive and, moreover, allows to consider control–affine systems with drift.

Since Theorem 2.6 holds true for every fixed integer N, it is natural to ask whether it is possible to realize not only the N-jet but also the whole diffeomorphism. In Chapter 3, Theorem 3.1 answers positively in the driftless case (i.e. $f_0 = 0$) and when M is a compact manifold. Moreover the proof is rather elementary and, again, it is based on the local invertibility of the map

$$\begin{array}{rccc} F: & C^\infty(U)^d & \rightarrow & \mathrm{Diff}_0(U) \\ & (a_1, \ldots, a_d) & \mapsto & e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d}\big|_U \,, \end{array} \tag{4}$$

for a given neighborhood of the origin $U \subset \mathbb{R}^d$ and given vector fields $X_1, \ldots, X_d$ linearly independent at 0. Then a geometric idea based on Orbit Theorem of Sussmann (see [39]) allows us to write a diffeomorphism in the image of F as the flow, at time 1, of a system of the form (1).
As a corollary we have that controllability of a system of vector fields on a compact connected manifold M implies a certain "controllability" on the group of diffeomorphisms $\mathrm{Diff}_0(M)$. Indeed, if $\mathcal{F}$ is a bracket generating family of vector fields, then, by Rashevsky–Chow Theorem, the system is completely controllable. That is, for every pair of points $q_0, q_1 \in M$ there exist $t_1, \ldots, t_k \in \mathbb{R}$ and $f_1, \ldots, f_k \in \mathcal{F}$ such that

$$q_0 = q_1 \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k},$$

and, by Theorem 3.1, for every diffeomorphism $P \in \mathrm{Diff}_0(M)$ there exist $a_1, \ldots, a_\ell \in C^\infty(M)$ and $g_1, \ldots, g_\ell \in \mathcal{F}$ such that

$$P = e^{a_1 g_1} \circ \cdots \circ e^{a_\ell g_\ell}.$$

In other words, if it is possible to join every two points of the manifold M by exponentials of vector fields in $\mathcal{F}$, then we can realize every diffeomorphism isotopic to the identity as composition of exponentials of vector fields in $\mathcal{F}$ rescaled by suitable smooth functions.

The second part of the thesis (Chapter 4 and 5) is mainly devoted in studying analytical properties of the exponential map. A first problem is to determine whether implicit function theorem applies. Indeed, in Chapter 2 the applicability of the implicit function theorem to the map (3) is crucial to allow small perturbation of this map to be surjective too. Differently from map (3), the domain and the image of map (4) are infinite dimensional Fréchet spaces, and, therefore, classical implicit function theorem does not apply. This is due to the "loss of derivatives". Indeed, while the exponential map sends C^k vector fields into C^k diffeomorphisms, its differential has an unbounded right inverse. In fact, the inverse maps the space C^k into C^{k-1}.
The problem of solving $F(x) = y$, near a given point x_0, in the case in which $F'(x_0)$ is invertible but with unbounded inverse has been of great interest in literature since the famous work of Nash [27] on isometric embedding in $\mathbb{R}^n$ of Riemannian manifolds. In this work Nash showed the basic idea for a technique that Jürgen Moser developed in [26] for a general approach to such problems. This method requires the invertibility of $F'(x)$ in an open neighborhood of x_0. The main idea of this technique is to replace the usual Picard iteration method used in the classical implicit function theorem with a modified Newton iteration scheme. The speed of convergence of this iteration scheme is sufficiently strong to compensate the divergences in the scheme due to the "loss of derivatives". The technique boasts a lot of extensions and applications showing its power and versatility. For example, we mention Sergeraert [36], that stated the theorem in terms of a category of maps between Fréchet spaces. Generalizations to implicit function theorems have been given by Zehnder [28, Chapter 6], and Hamilton [17, 18]. A great number of applications have been made in almost every branch of mathematics. To cite just a few, we mention the applications made by Nash [27], Jacobowitz [21], and Gromov [16] to isometric embeddings, by Moser and Zehnder [41] to small divisor problems , by Hörmander [20] to problems in gravitation, by Beale [6] to water waves, by Schaeffer [33, 34] to free boundary problems in electromagnetics, by Sergeraert [37] to catastrophe theory, and by Hamilton [19] to foliations.

Depending on the applications one has in mind there are many way to state the generalized implicit function theorems, usually called Nash–Moser implicit function theorems. In Chapter 4 we present two statements. First we present a Hamilton's version (see [19]). This result is used in Chapter 5 to show not only that the product of exponential (4) is locally onto but also that its differential is surjective. Moreover an explicit formula for the inverse of the differential is provided. We conclude in Section 5.3 with the conjecture that this property allows small perturbations of the map to be surjective, like in the finite dimensional case of Chapter 2. In order to achieve this result we present also a Zehnder version (see [28, Chapter 6]) of the generalized implicit function theorem that does not require the differential to be invertible in an open set but just the existence of an “approximate right inverse”. Finally, we present this result as an application of a conjugacy problem by Moser (see [26]) that shows how the iteration scheme works under the weaker hypotesis of existence of an “approximate right inverse”.

CHAPTER 1

Preliminaries and geometric control theory

The present chapter is mostly an introduction to the language employed in the thesis and a brief presentation of geometric control theory. It contains definitions and results in geometric control theory and its related areas of mathematics, such as differential geometry and functional analysis. The topics treated in this chapter are only sketched and for a deep study in control theory we refer to two cornerstones as [4] and [23]. We took also inspiration from the lecture notes [22]. The result of Section 1.9 firstly appears in our paper [2].

1.1 Control systems

Let M be a smooth d-dimensional connected real manifold. Throughout this text smooth means C^∞. By a *control system* we mean a system of the form

$$\dot{q} = f_u(q), \quad q \in M,\ u \in U\,, \tag{1.1}$$

where q, called *state*, takes values in the manifold M and u, called *control*, takes values in a set U that is an arbitrary (usually closed) subset of $\mathbb{R}^m$. We call M the *state space* of the system and U the *control set* or *space of control parameters*. For every u fixed f_u is a smooth vector field in $\operatorname{Vec} M$ and the system equation $\dot{q} = f_u(q)$ defines a single dynamical system. A control system can be seen as a family of dynamical systems or, similarly, as a dynamical system whose dynamical laws are not fixed as they are in the problems of classic physics but they depend on the control parameters with which one can control the behavior of the system. The basic challenge is to understand the effects of the controls on the dynamics of the system.

The control system (1.1) is often written in the form

$$\dot{q} = f(q,u), \quad q \in M,\ u \in U\,.$$

This notation is useful when the control u is a function. It is important to point out that for every $u \in U$, $f(\cdot,u)$ is a single object, a vector field on M.

If the control u is a curve in the space of control parameters, namely a measurable and locally bounded function $u : \mathbb{R} \to U$, then u is called *classical control*. The corresponding trajectories are integral curves of the time–varying vector field $q \to f(q, u(t))$.

A control u is called *feedback control* if u is a smooth function of the state $q \in M$. The corresponding system $q \to f(q, u(q))$ is called *closed–loop* system.

Finally, a control can be a combination of both types, that is, a mapping $u : \mathbb{R} \times M \to U$. Such a control is called *time–varying feedback control*.

In the following we deal also with special classes of nonlinear systems. A *control–affine system* is a control system of the form

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i, \quad q \in M\,, \tag{1.2}$$

where $U = \mathbb{R}$. The vector field f_0 is called *drift*. We say that a system is *driftless* or *homogeneous (with respect to control)* if it is of the form (1.2) with $f_0 = 0$.

1.2 Operators on the algebra of smooth functions

There is a natural way to associate to points, diffeomorphisms, and vector fields on the manifold M operators on the algebra $C^\infty(M)$ of real valued smooth function on the manifold M. We define the operations of addition, multiplication, and multiplication by a scalar on $C^\infty(M)$ pointwise. That is, if $a, b \in C^\infty(M)$, $\alpha \in \mathbb{R}$, and $q \in M$ then

$$\begin{aligned} (a+b)(q) &= a(q) + b(q)\,, \\ (ab)(q) &= a(q)b(q)\,, \\ (\alpha a)(q) &= \alpha a(q)\,. \end{aligned}$$

The space $C^\infty(M)$ endowed with the operations above is a commutative algebra on $\mathbb{R}$.

We associate to every point $q \in M$ the evaluation operator

$$\begin{array}{rccc} q: & C^\infty(M) & \to & \mathbb{R} \\ & a & \mapsto & a(q), \end{array}$$

that is a homomorphism of the algebras $C^\infty(M)$ and $\mathbb{R}$.

If V is a smooth vector field and a a smooth function on M, then Va denotes the smooth function $q \to V(q)(a)$, namely the derivative of a along the direction $V(q)$. To a smooth vector field V then corresponds a derivation on the algebra $C^\infty(M)$, that is a linear operator

$$V : C^\infty(M) \to C^\infty(M)\,,$$

that satisfies Leibniz rule, namely

$$V(ab) = V(a)b + aV(b), \quad a, b \in C^\infty(M)\,.$$

Finally, any diffeomorphism P on M naturally defines an automorphism

$$P : C^\infty(M) \to C^\infty(M)\,,$$

of the algebra $C^\infty(M)$ as follows:

$$(Pa)q = a(P(q)), \quad q \in M,\ a \in C^\infty(M)\,.$$

1.3 Families of vector fields and exponentials

It is natural to expect that basic properties of a control system depend on interconnections between the different dynamical systems corresponding to different controls. We represent our dynamical systems (1.1) by the family of vector fields $\mathcal{F} = \{f_u : u \in U\}$. This allows us to work with the geometric structure of the control system in a coordinate independent way.

We denote by $\operatorname{Vec} M$ the Lie algebra of smooth vector fields on M. In order to define the Lie bracket we need to look at smooth vector fields as derivations on the algebra $C^\infty(M)$.

Definition 1. Given $V, W \in \operatorname{Vec} M$, their *Lie bracket* $[V, W]$ is defined by

$$[V, W]a = W(Va) - V(Wa)\,, \qquad \text{for } a \in C^\infty(M)\,.$$

Definition 2. Given $V, W \in \operatorname{Vec} M$. We say that V and W *commute* if $[V, W] = 0$.

The Lie bracket of two vector fields is another vector field which, roughly speaking, measures noncommutativeness of the flows of both vector fields.

The space of vector field $\operatorname{Vec} M$ endowed with the product given by the Lie bracket is a Lie algebra. Given a family of vector fields $\mathcal{F}$ we denote by $\operatorname{Lie}(\mathcal{F})$ the Lie algebra generated by vector fields in $\mathcal{F}$. Namely, $\operatorname{Lie}(\mathcal{F})$ is the smallest vector subspace $\mathcal{V}$ of $\operatorname{Vec} M$ such that $[f, \mathcal{V}] \subset \mathcal{V}$ for every $f \in \mathcal{F}$. In general $\operatorname{Lie}(\mathcal{F})$ is an infinite–dimensional subspace of $\operatorname{Vec} M$.
Given $q \in M$, $\operatorname{Lie}_q \mathcal{F}$ denotes the algebra of tangent vector $f(q)$ with $f \in \operatorname{Lie}(\mathcal{F})$. It follows that $\operatorname{Lie}_q \mathcal{F}$ is a linear subspace of T_qM and hence it is finite–dimensional.

Definition 3. If a family $\mathcal{F}$ is such that

$$\operatorname{Lie}_q \mathcal{F} = T_qM \qquad \text{for every } q \in M\,, \tag{1.3}$$

we say that the family is *bracket generating* or *completely nonholonomic*. Condition (1.3) is sometimes called *Hörmander condition*.

Definition 4. A vector field $V \in \mathrm{Vec}\, M$ is said to be *complete* if, for every $q_0 \in M$, the solution of the Cauchy problem

$$\begin{cases} \dot{q}(t) &= V(q(t)) \\ q(0) &= q_0\,, \end{cases} \tag{1.4}$$

is defined for every $t \in \mathbb{R}$.

We call *nonautonomous vector field* a path V_t on $\mathrm{Vec}\, M$

$$t \mapsto V_t, \quad V_t \in \mathrm{Vec}\, M,\ t \in \mathbb{R}\,,$$

locally integrable with respect to $t \in \mathbb{R}$.

Definition 5. A nonautonomous vector field V_t is said to be *complete* if, for every $t_0 \in \mathbb{R}$ and $q_0 \in M$, the solution of the Cauchy problem

$$\begin{cases} \dot{q}(t) &= V_t(q(t)) \\ q(t_0) &= q_0\,, \end{cases} \tag{1.5}$$

is defined for every $t \in \mathbb{R}$.

If M is a compact manifold then every vector field is complete. When $M = \mathbb{R}^d$ we assume that every vector fields under consideration satisfies the growth condition $V_t(q) \leq \phi(t)(1 + |q|)$, where ϕ is a locally integrable function. Under these assumptions every vector field in this thesis can be supposed complete without loss of generality.

Let $V \in \mathrm{Vec}\, M$, then the map which associates with any $q_0 \in M$ the value of the solution of

$$\begin{cases} \dot{q}(t) &= V(q(t)) \\ q(0) &= q_0\,, \end{cases}$$

evaluated at a fixed time t, is a diffeomorphisms from M into itself, denoted by

$$e^{tV} : q_0 \mapsto e^{tV}(q_0)\,,$$

and called the *flow of* V *at time* t. If $t = 1$ we also call this map the *exponential of* V.

Similarly, if V_τ is nonautonomous vector field, then the map which associates with any $q_0 \in M$ the value of the solution at a fixed time t of system

$$\begin{cases} \dot{q}(t) &= V_t(q(t)) \\ q(t_0) &= q_0\,, \end{cases}$$

is called *(right) chronological exponential* of V_τ and it is denoted by

$$\overrightarrow{\exp} \int_{t_0}^{t} V_\tau \, d\tau : M \to M \,. \tag{1.6}$$

The map

$$V \in \mathrm{Vec}\, M \mapsto e^V \in \mathrm{Diff}_0(M)$$

that associates to every vector field its exponential is called *exponential map*. When it does not create ambiguity we will call exponential map also a map that associates to a nonautonomous vector field its chronological exponential at a fixed time, or, given a control system, that associates the control to the flow, at time fixed, of the system.

1.4 Action of diffeomorphisms on vector fields

Every diffeomorphism $P \in \mathrm{Diff}(M)$ naturally defines the following transformation of a vector field V:

$$\mathrm{Ad}\, PV(p) = P \circ V \circ P^{-1}, \quad q \in M \,.$$

In fact, $\mathrm{Ad}\, P$ is the linear operator on $\mathrm{Vec}\, M$ corresponding to a change of coordinates P. The coordinate change $p = P(q)$ transforms the differential equation $\dot{p} = V(p)$ into the equation $\dot{q} = \tilde{V}(q)$ where $\tilde{V} = \mathrm{Ad}\, PV$. Another notation used to denote the action of the group of diffeomoprhisms $\mathrm{Diff}(M)$ on the algebra of vector fields $\mathrm{Vec}\, M$ is

$$P_* = \mathrm{Ad}\, P^{-1} \,.$$

The operation $\mathrm{Ad}\, P$ is a linear operator on the space of vector fields on M. Namely, let V, W be vector fields on M and $\lambda, \mu \in \mathbb{R}$ then

$$\mathrm{Ad}\, P(\lambda V + \mu W) = \lambda \,\mathrm{Ad}\, P(V) + \mu \,\mathrm{Ad}\, P(W) \,.$$

If Q is another diffeomorphism on M, then

$$\mathrm{Ad}(P \circ Q)V = \mathrm{Ad}\, P \,\mathrm{Ad}\, QV \,.$$

Moreover an easy computation gives

$$\begin{aligned} \mathrm{Ad}\, P[V, W] &= P \circ V \circ W \circ P^{-1} - P \circ W \circ V \circ P^{-1} \\ &= [\mathrm{Ad}\, PV, \mathrm{Ad}\, PW] \,. \end{aligned}$$

Now let $P^t = e^{tV}$ be the flow of V, then we have

$$\begin{aligned} \frac{d}{dt}(\mathrm{Ad}\, P^t W)\big|_{t=0} &= \frac{d}{dt} P^t \circ W \circ (P^t)^{-1}\big|_{t=0} \\ &= V \circ W - W \circ V \\ &= [V, W] \,. \end{aligned} \tag{1.7}$$

We set

$$\operatorname{ad} V = \frac{d}{dt} \operatorname{Ad} P^t\big|_{t=0} ,$$

then, $\operatorname{ad} V$ is the linear operator on the algebra $\operatorname{Vec} M$ that satisfies

$$(\operatorname{ad} V)W = [V, W] .$$

Let $V, W \in \operatorname{Vec} M$, a be a smooth function on M, and P a diffeomorphism, then the following identities hold

$$\begin{aligned} (\operatorname{ad} V)(aW) &= (Va)W + a(\operatorname{ad} V)W , \\ (\operatorname{Ad} P)aV &= (Pa) \operatorname{Ad} PV . \end{aligned}$$

1.5 Basic elements of chronological calculus

The chronological calculus is a mathematical tool that allows intrinsic description and manipulation of nonlinear objects and dynamics. It has been first developed by Agrachev and Gamkrelidze in [3]. In this section we give some simple properties related to this tool that we use in the sequel. In this section V_τ and W_τ denote nonautonomous vector fields

First, note that it follows from the definition that the chronological exponential satisfies the differential equation

$$\frac{d}{dt} \overrightarrow{\exp} \int_0^t V_\tau \, d\tau = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau \circ V_t .$$

Moreover, in the sequel we use the following simple properties of the chronological exponential

$$\begin{aligned} \overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau \circ \overrightarrow{\exp} \int_{t_1}^{t_2} V_\tau \, d\tau &= \overrightarrow{\exp} \int_{t_0}^{t_2} V_\tau \, d\tau , \\ \overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau &= \left(\overrightarrow{\exp} \int_{t_1}^{t_0} V_\tau \, d\tau \right)^{-1} . \end{aligned}$$

By the relation (1.7) it follows

$$\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau = \operatorname{Ad} \overrightarrow{\exp} \int_0^t V_\tau \, d\tau . \tag{1.8}$$

Thus, if $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ then the family of operators $\operatorname{Ad} P^t$ satisfies the ODE

$$\begin{cases} \dfrac{d}{dt} \operatorname{Ad} P^t &= \operatorname{Ad} P^t \circ \operatorname{ad} V_t , \\ \operatorname{Ad} P^0 &= \operatorname{Id} . \end{cases}$$

In the case of an autonomous vector field V relation (1.8) is as follows

$$e^{t\,\mathrm{ad}\,V} = \mathrm{Ad}\, e^{tV} .$$

We now show a simple identity, called *variation formula*, describing the flow of a perturbed vector field. Let $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$, the flow of the vector field $V_\tau + W_\tau$ can be written as follows:

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau)\, d\tau = \overrightarrow{\exp} \int_0^t (\mathrm{Ad}\, P^\tau) W_\tau \, d\tau \circ P^t . \tag{1.9}$$

Namely, the perturbed flow is a composition of the flow P^t with the flow of the perturbation W^t twisted by P^t. Similarly, for autonomous vector fields V, W we have

$$e^{t(V+W)} = \overrightarrow{\exp} \int_0^t e^{\tau\,\mathrm{ad}\,V} W \, d\tau \circ e^{tV} .$$

If $V_t(s)$ is a nonautonomous vector field smoothly depending on a parameter s, then from variation formula easy follows the identity below, very useful to compute the differential of the exponential map. Let $P^t(s) = \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau$ then

$$\frac{\partial}{\partial s} P^t(s) = \int_0^t (\mathrm{Ad}\, P^\tau(s)) \frac{\partial}{\partial s} V_\tau(s)\, d\tau \circ \ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau . \tag{1.10}$$

1.6 The C^∞ topology

Here we focus on the space of smooth functions on a manifold and its topology. The topics of this section can be found in every text of functional analysis, in particular we refer to [32].

We denote by $C^\infty(M)$ the space of infinitely differentiable functions from the manifold M to $\mathbb{R}$. We define a topology on $C^\infty(M)$ which turns $C^\infty(M)$ into a Fréchet space. By Whitney's Theorem, a smooth manifold M can be properly embedded into $\mathbb{R}^N$ for N sufficiently large. Consider the constant vector fields $\frac{\partial}{\partial x_i}$, for $i = 1, \ldots, N$, of the basis of $\mathrm{Vec}\,\mathbb{R}^N$. Denote by h_i, for $i = 1, \ldots, N$ the vector field on M that is the orthogonal projection, from $\mathbb{R}^N$ to M, of $\frac{\partial}{\partial x_i}$. Therefore, $h_1, \ldots, h_N$ span the tangent space T_qM at each point $q \in M$. Now, consider a sequence of compact subsets K_n of M such that

$$K_n \subset K_{n+1} , \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} K_n = M .$$

Then, we define the family of seminorms $\|\cdot\|_{C^n(K_n)}$ on $C^\infty(M)$ as follows

$$\|a\|_{C^n(K_n)} = \sup\{ |h_{i_1} \circ \cdots \circ h_{i_k} a(q)| \, : \, q \in K_n, 0 \le k \le n, i_j \in \{1, \ldots, N\} \} .$$

This family of seminorms defines the topology of $C^\infty(M)$. A local base for this topology is given by the sets

$$\left\{a \in C^\infty(M) \,:\, \|a\|_{C^n(K_n)} < \frac{1}{n}\right\}.$$

We refer to the topology above as the C^∞ *topology*, or standard topology.
Thus $C^\infty(M)$ is a Fréchet space, that is a complete metrizable locally convex topological vector space. Moreover, the space of smooth functions carries a more refined structure being, in fact, the intersection of the Banach spaces $C^k(M)$, namely

$$C^\infty(M) = \bigcap_{k=1}^{\infty} C^k(M)\,, \quad \text{where} \quad C^k(M) \subset C^{k-1}(M)\,.$$

This fact leads, in Section 4.2, to the definition of *tame space*.
It is important to remark that $C^\infty(M)$ is *not normable*.
Note that if M is compact (or precompact) then we can take $K_n = M$ for every n, is such a case we set $\|\cdot\|_n := \|\cdot\|_{C^n(M)}$.

If B is a neighborhood of the origin in $\mathbb{R}^d$ then we call $C_0^\infty(B)$ the closed subspace of real smooth functions from B to $\mathbb{R}$ that vanish at the origin.

We define the seminorms for a vector field $V \in \operatorname{Vec} M$ as

$$\|V\|_n = \sup\{\|Va\|_n \,:\, \|a\|_{n+1} = 1\}\,.$$

Then, we endow also $\operatorname{Vec} M$ and $\operatorname{Diff}(M)$ with the C^∞ topology. As for $C^\infty(M)$, we will show in Section 4.2, that also these spaces carry the more refined structure of tame space.

We conclude with the definition of seminorms for paths in the group of diffeomorphisms $\{P^t\}_{t\in[0,1]} \subset \operatorname{Diff}(M)$ and for nonautonomous vector fields $\{V_t\}_{t\in[0,1]} \subset \operatorname{Vec} M$. We set

$$\|\{P^t\}_t\|_n = \sup_{t\in[0,1]} \|P^t\|_n\,, \quad \text{and} \quad \|\{V_t\}_t\|_n = \sup_{t\in[0,1]} \|V_t\|_n\,.$$

When it does not create ambiguity we replace the notation $\|\{\ \}_t\|_n$ with the more readable $\|\cdot\|_n$.

1.7 The group of diffeomorphisms

We denote by $\operatorname{Diff}(M)$ the set of all diffeomorphisms of M. $\operatorname{Diff}(M)$ is an infinite dimensional Lie group. Indeed, it is an infinite dimensional manifold modeled on the Fréchet space of smooth functions from $\mathbb{R}^d$ to $\mathbb{R}^d$. Moreover, it is a group with respect to the composition "$\circ$" with null element the identical diffeomorphism Id. Finally the operation $(P, Q) \mapsto P^{-1} \circ Q$ is smooth.

We denote by $\mathrm{Diff}_0(M)$ the connected component of Id of $\mathrm{Diff}(M)$.

The tangent space to $\mathrm{Diff}(M)$ at the identity is the Lie algebra of vector fields, i.e.

$$T_{\mathrm{Id}}\,\mathrm{Diff}(M) = \mathrm{Vec}\,M\,.$$

Indeed, let $P \in \mathrm{Diff}_0(M)$. Consider a path $P^t \in \mathrm{Diff}_0(M)$ such that $P^1 = P$ and $P^0 = \mathrm{Id}$, then consider the nonautonomous vector field

$$V_t = \left(P^t\right)^{-1} \circ \frac{d}{dt}P^t\,,$$

which is such that

$$P = \overrightarrow{\exp}\int_0^1 V_t\,dt\,.$$

Thus every diffeomorphism is the exponential of a nonautonomous vector field and, in particular,

$$\frac{d}{dt}P^t\big|_{t=0} \in \mathrm{Vec}\,M\,,$$

for every path $P^t \in \mathrm{Diff}_0(M)$ such that $P^0 = \mathrm{Id}$. In other words, we showed that $\mathrm{Vec}\,M$ is the *Lie algebra* of the Lie group $\mathrm{Diff}(M)$.

In the case $M = \mathbb{R}^d$ then any orientation preserving diffeomorphism of $\mathbb{R}^d$ is isotopic to the identity. Indeed, let P be an orientation preserving diffeomorphisms of $\mathbb{R}^d$. We can suppose without loss of generality that P fixes the origin just taking the isotopy $H^1(t,\cdot) = P - (1-t)P(0)$. Now, rename for simplicity $P := H^1(0,\cdot)$ and consider another isotopy

$$H^2(t,q) = P(tq)/t, \quad t \in (0,1], \qquad \text{and} \qquad H^2(0,q) = \lim_{t\to 0} P(tq)/t.$$

Since P is orientation preserving then $H^2(0,\cdot)$ belongs to the connected component of the identity of the group of linear invertible operators on $\mathbb{R}^d$, $GL^+(d,\mathbb{R})$.

Definition 6. Given $P \in \mathrm{Diff}\,M$, we define the *support* of P as follows

$$\mathrm{supp}\,P = \overline{\{q \in M : P(q) \neq q\}}\,.$$

We conclude with two definitions of subgroups of $\mathrm{Diff}(M)$ generated by exponentials of vector fields in a family. Let $\mathcal{F} \subset \mathrm{Vec}\,M$ be a family of vector fields we set

$$\mathrm{Gr}\mathcal{F} = \{e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \,:\, t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\}\,.$$

Moreover, we define the group generated by exponentials of vector fields in $\mathcal{F}$ rescaled by smooth functions as follows

$$\mathrm{Gr}_S\mathcal{F} = \{e^{a_1 f_1} \circ \cdots \circ e^{a_k f_k} \,:\, a_i \in C^\infty(M), f_i \in \mathcal{F}, k \in \mathbb{N}\}\,.$$

One of the purposes of this thesis is of study under what conditions on $\mathcal{F}$ these subgroup are the whole group $\mathrm{Diff}_0(M)$.

1.8 Attainable sets

The basic challenge in control theory is to study what are the points of the manifold reachable from a starting point by choosing various type of controls. The set of points reachable is called *attainable set* or *reachable set* and, given a control system

$$\dot{q} = f(u, q), \quad q \in M, u \in U\,, \tag{1.11}$$

is defined as follows.

Definition 7. The *attainable set* of the control system (1.11) with piecewise constant controls $u(t)$, from a point $q_0 \in M$ for a time $t \geq 0$ is

$$\mathcal{A}_{q_0}(t) = \{P^t_{u(\cdot)}(q_0)\ :\ u \text{ piecewise constant }\} \subset M\,,$$

where P^t_u is the flow of system (1.11) associated to the control u.

We define also the attainable set from $q_0 \in M$ for arbitrary nonnegative time, that is

$$\mathcal{A}_{q_0} = \bigcup_{t \geq 0} \mathcal{A}_{q_0}(t)\,.$$

One of purposes of this thesis is to study the set of diffeomorphisms that are flow, at time fixed, of the control system (1.11). We define the set of attainable diffeomorphisms as follows.

Definition 8. The set of attainable diffeomorphisms of the control system (1.11) with time–varying feedback controls $u(t, q)$ for a time $t \geq 0$ is

$$\mathbf{A}_t = \{P^t_u\ :\ u \text{ time–varying feedback control }\} \subset \mathrm{Diff}_0(M)\,,$$

where P^t_u is the flow of system (1.11) associated to the control u.

Given a family of vector field $\mathcal{F}$ the definition of attainable set for arbitrary nonnegative time is

$$\mathcal{A}_{q_0} = \{q_0 \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k}\ :\ t_i \geq 0, f_i \in \mathcal{F}, k \in \mathbb{N}\}\,.$$

We also call $\mathcal{A}_{q_0}$ the attainable set of $\mathcal{F}$ by piecewise constant controls and, if necessary, we highlight the dependence of $\mathcal{A}_{q_0}$ on the family $\mathcal{F}$ using the notation $\mathcal{A}_{q_0}(\mathcal{F})$. Consider now a larger set, the orbit of the family through a point q_0:

$$\begin{aligned}\mathcal{O}_{q_0} &= \{q_0 \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k}\ :\ t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\} \\ &= \{q_0 \circ P\ :\ P \in \mathrm{Gr}\mathcal{F}\}\,.\end{aligned}$$

Note that, in general, $\mathcal{A}_{q_0} \subset \mathcal{O}_{q_0}$. Indeed in the orbit we can move along integral curves of $\mathcal{F}$ both forward and backward, while in the attainable set are contained only trajectories for positive time. If a family $\mathcal{F}$ is symmetric, namely if $\mathcal{F} = -\mathcal{F}$, then the attainable sets coincide with the orbits, i.e. $\mathcal{A}_{q_0} = \mathcal{O}_{q_0}$.

A basic properties of families of vector fields is that their orbits are manifolds. This is the statement of the "Orbit Theorem" that marks a point of departure for geometric control theory.

Theorem 1.1 (Orbit Theorem, Sussmann)**.** *The orbit of $\mathcal{F}$ through each point q is a connected submanifold of M. Moreover,*

$$T_p\mathcal{O}_q = \mathrm{span}\{q \circ \mathrm{Ad}\, Pf \;:\; P \in \mathrm{Gr}\mathcal{F}, f \in \mathcal{F}\}, \quad p \in \mathcal{O}_q\,.$$

The importance of this result for control theory comes also from the following result that gives sufficient condition for controllability. Indeed, this classical result, although independent, can be seen as a corollary of the Orbit Theorem.

Theorem 1.2 (Chow – Rashevsky)**.** *Let $\mathcal{F}$ be a bracket generating family of vector fields. Then*

$$\mathcal{O}_q = M, \qquad \textit{for any } q \in M\,.$$

As a consequence, we have that if $\mathcal{F}$ is a bracket generating family, then $\mathrm{Gr}\mathcal{F}$ acts transitively on M. Namely, for every pair of points $q_0, q_1 \in M$ there exist an element of $P \in \mathrm{Gr}\mathcal{F}$ such that $q_0 = P(q_1)$. There is a classical result, due to Lobry [25], which claims that $\mathrm{Gr}\{f_1, f_2\}$ acts transitively on M for a generic pair of smooth vector fields (f_1, f_2), i.e. the set of pairs of vector fields (f_1, f_2) such that $\mathrm{Gr}\{f_1, f_2\}$ acts transitively on M is an open dense subset of the product space $\mathrm{Vec}\, M \times \mathrm{Vec}\, M$.

Definition 9. We say that a system (or a family of vector fields) is *completely controllable* if $\mathcal{A}_q = M$, for every $q \in M$.

Hence, a system is completely controllable if there exists an admissible control, and therefore an admissible trajectory, which drives any given point of the manifold to any other point. As a corollary of Chow–Rashevsky Theorem we have the following.

Corollary 1.3. *A symmetric bracket–generating family on a connected manifold is completely controllable.*

The remark below, on the product of exponentials, directly follows from the Orbit Theorem.

Remark 1. If $\mathcal{F}$ be a bracket generating family of vector fields, then by Chow–Rashevsky Theorem $\mathcal{O}_q = M$ for every $q \in M$ and, therefore, by Orbit Theorem, for every $q \in M$, we have

$$T_qM = \mathrm{span}\{q \circ \mathrm{Ad}\, Pf \;:\; P \in \mathrm{Gr}\mathcal{F}, f \in \mathcal{F}\}\,.$$

If $X_1, \ldots, X_d$ are such that $\mathrm{span}\{X_1(q), \ldots, X_d(q)\} = T_qM$, then $X_i = \mathrm{Ad}\, P^i f_i$ with $P^i \in \mathrm{Gr}\mathcal{F}$ and $f_i \in \mathcal{F}$. This fact will be very useful in what follows since allows us to prove that if $a_1, \ldots, a_d \in C^\infty(M)$ then

$$e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d} = P_1 \circ e^{(a_1 \circ P_1) f_1} \circ P_1^{-1} \circ \cdots \circ P_d \circ e^{(a_d \circ P_d) f_d} \circ P_d^{-1},$$

belongs to $\mathrm{Gr}_S\mathcal{F}$. In other words the image of the map

$$F : (a_1, \ldots, a_d) \mapsto e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d},$$

belongs to the group $\mathrm{Gr}_S\mathcal{F}$.

Finally we state an important Theorem by Krener [24] that gives a description of attainable sets for bracket generating systems.

Theorem 1.4 (Krener)**.** *If $\mathcal{F} \subset \mathrm{Vec}\, M$ is a bracket generating family of vector fields, then*

$$\mathcal{A}_q \subset \overline{\mathrm{int}\mathcal{A}_q}\,, \quad \textit{for every } q \in M\,.$$

In particular the theorem states that attainable sets for arbitrary nonnegative time have nonempty interior. Moreover, they are full-dimensional and they cannot have boundary points isolated from interior points. On the other hand, attainable sets may be open sets, manifolds with smooth boundary, and manifold with boundary having singularities. A remarkable corollary of Krener's theorem is the following.

Corollary 1.5. *Let $\mathcal{F} \subset \mathrm{Vec}\, M$ be a bracket generating family of vector fields. If $\overline{\mathcal{A}_q} = M$, for some $q \in M$, then $\mathcal{A}_q = M$.*

The sense of this corollary is that, when studying controllability for a bracket generating system, it is possible to replace the attainable set by its closure. If adding a vector field to a family does not change the closure of the attainable set then we say that the vector field is compatible with the system.

Definition 10. We say that a vector field $f \in \mathrm{Vec}\, M$ is *compatible* with a system $\mathcal{F} \subset \mathrm{Vec}\, M$ if

$$\mathcal{A}_q(\mathcal{F} \cup f) \subset \overline{\mathcal{A}_q(\mathcal{F})}\,.$$

1.9 Relaxation

In this last section of this chapter we prove that, if $\mathcal{F}$ is a bracket generating family of vector fields, then $\mathrm{Gr}_S\mathcal{F}$ is dense in $\mathrm{Diff}_0\mathbb{R}^d$. Although this result is stated on $\mathbb{R}^d$, it can be easily extended to a manifold M. The proof makes use of simple relaxation arguments and, in particular, of the following well-known fact (see, for instance, [4, Lemma 8.2]).

Lemma 1.6. *Let Z_t and Z_t^n for $n = 1, 2, \dots$ and $t \in [0,1]$ be nonautonomous vector fields on M. If*

$$\int_0^t Z_\tau^n \, d\tau \to \int_0^t Z_\tau \, d\tau, \quad \text{as } n \to \infty,$$

in the standard C^∞ topology and uniformly with respect to $t \in [0,1]$, then

$$\overrightarrow{\exp} \int_0^t Z_\tau^n \, d\tau \to \overrightarrow{\exp} \int_0^t Z_\tau \, d\tau, \quad \text{as } n \to \infty,$$

in the same topology.

Another Lemma needed in the proof is the following result. In particular it states that there exists a time–varying vector field, which is piecewise constant in time, whose exponential is arbitrary close to a given path on the group of diffeomorphisms.

Lemma 1.7. *Let $X_1, \dots, X_k$ be smooth vector fields on $\mathbb{R}^d$ and $\mathcal{A}$ be a closed subspace of $C^\infty(\mathbb{R}^d)$. Then, for any time-varying vector field of the form*

$$V_t = \sum_{i=1}^k a_i(t, \cdot) X_i \,,$$

where $a_i(t,\cdot) \in \mathcal{A}$ and $0 \le a_i(t,q) \le \varphi(t)$ for some locally integrable φ, $i = 1, \dots, k$, there exists a sequence of time-varying, piecewise constant with respect to t, vector fields Z_t^n such that

$$Z_t^n \in \{ a X_i \mid a \in \mathcal{A}, i = 1, \dots, k \}, \quad \text{for any } t \in [0,1]$$

and

$$\overrightarrow{\exp} \int_0^t Z_\tau^n \, d\tau \longrightarrow \overrightarrow{\exp} \int_0^t V_\tau \, d\tau, \quad \text{as } n \to \infty$$

in the standard topology and uniformly with respect to $t \in [0,1]$.

Proof. First, note that we can suppose, without loss of generality, that $a_i(t,\cdot)$ is piecewise constant in t for every $i = 1, \dots, k$. Indeed, for any $i = 1, \dots, k$, the sequence

$$a_i^n(t,q) = n \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} a_i(\tau, q) \, d\tau \, \chi_j^n(t), \tag{1.12}$$

where $\chi_j^n(t)$ is the characteristic function of the interval $[\frac{j-1}{n}, \frac{j}{n}]$, is such that

$$\int_0^t \sum_{i=1}^k a_i^n(\tau, \cdot) X_i \, d\tau \to \int_0^t V_\tau \, d\tau, \quad \text{as } n \to \infty,$$

uniformly with respect to t and in the C^∞-topology. Therefore Lemma 1.6 allows us to suppose that $a_i(t,\cdot)$ is piecewise constant in t for every i.

Let ℓ be a positive integer such that V_t is constant on $[\frac{j-1}{\ell},\frac{j}{\ell}]$ for every $j = 1,\dots,\ell$. We can write

$$a_i(t,q) = \sum_{j=1}^{\ell} a_i^j(q)\chi_j^\ell(t)\,, \tag{1.13}$$

with $a_i^j(q) \geq 0$ for every $q \in \mathbb{R}^d$. Let

$$\alpha^j = \sum_{i=1}^{k} a_i^j\,, \tag{1.14}$$

and let $\{\varepsilon_n\}$ a sequence of nonnegative smooth functions of $\mathbb{R}^d$ such that $\varepsilon_n(0) = 0$ for every n and $\varepsilon_n \to 0$ as $n \to \infty$ in the C^∞-topology. Then $\alpha_n^j = \alpha^j + \varepsilon_n$ is strictly positive on $\mathbb{R}^d \setminus \{0\}$ for every j and n.

Now, for every positive integer n and $j = 1,\dots,\ell$, let $b_n^{j,i} = a_i^j/\alpha_n^j$ and consider the family of intervals:

$$A_n^{j,i} = \bigcup_{m=0}^{n-1}\left[\frac{j-1}{\ell}+\frac{m}{n\ell}+\frac{b_n^{j,1}+\cdots+b_n^{j,i-1}}{n\ell}\,,\ \frac{j-1}{\ell}+\frac{m}{n\ell}+\frac{b_n^{j,1}+\cdots+b_n^{j,i}}{n\ell}\right),$$

for $i = 2,\dots,k$, and

$$A_n^{j,1} = \bigcup_{m=0}^{n-1}\left[\frac{j-1}{\ell}+\frac{m}{n\ell}\,,\ \frac{j-1}{\ell}+\frac{m}{n\ell}+\frac{b_n^{j,1}}{n\ell}\right).$$

The sequence of vector fields

$$Z_t^n = \alpha_n^j X_i, \quad \text{if } t \in A_n^{j,i}, \tag{1.15}$$

is such that

$$\int_0^t Z_\tau^n\,d\tau \to \int_0^t V_\tau\,d\tau, \quad \text{as } n \to \infty.$$

In the standard topology and uniformly with respect to $t \in [0,1]$. The statement then follows from Lemma 1.6. □

In particular, a consequence of last lemma is that, given a system $\mathcal{F} \subset \operatorname{Vec} M$, $f, g \in \mathcal{F}$, and $a, b \in C^\infty(M)$, then the vector field $af + bg$ is compatible with $\mathcal{F}$.

The main result of this section is as follows.

Proposition 1.8 (Approximation)**.** *Let $\mathcal{F} \subseteq \operatorname{Vec}\mathbb{R}^d$ be a bracket generating family of vector fields on $\mathbb{R}^d$ such that*

$$af \in \mathcal{F} \text{ for any } a \in C^\infty(\mathbb{R}^d), f \in \mathcal{F}. \tag{1.16}$$

Then, for any orientation preserving diffeomorphism P of $\mathbb{R}^d$, there exists a sequence $\{P_n\}_n \subset \mathrm{Gr}\mathcal{F}$ such that

$$P_n \longrightarrow P, \qquad \text{as } n \to \infty\,,$$

in the standard topology.

Proof. Let $P^t \subset \mathrm{Diff}_0(\mathbb{R}^d)$ be a path such that $P^0 = \mathrm{Id}$ and $P^1 = P$. And consider the time–varying vector field V_t such that $\overrightarrow{\exp} \int_0^t V_\tau \, d\tau = P^t$.

Recall that, since $\mathcal{F}$ is bracket generating then, by Remark 1, it is possible to take $X_1, \ldots, X_d$ such that $X_i = P^i_* f_i$ with $P^i \in \mathrm{Gr}\mathcal{F}$, $f_i \in \mathcal{F}$, $i = 1, \ldots, d$, and

$$V_t = \sum_{i=1}^d a_i(t,\cdot) X_i,$$

where $a_i(t,\cdot) \in C^\infty(\mathbb{R}^d)$ for any $t \in [0,1]$.

By Proposition 1.7 there exists a sequence $Z^n_t \in \{\alpha X_i \mid \alpha \in C^\infty(\mathbb{R}^d), i = 1, \ldots, d\}$ such that

$$\overrightarrow{\exp} \int_0^t Z^n_\tau \, d\tau \to P^t, \quad \text{as } n \to \infty,$$

and the convergence is uniform with respect to $t \in [0,1]$.
Let $P_n := \overrightarrow{\exp} \int_0^1 Z^n_t \, dt$, then

$$P_n \to P, \quad \text{as } n \to \infty.$$

It remains to prove that $P_n \in \mathrm{Gr}\mathcal{F}$ for every n. Since Z^n_t is piecewise constant in t, so, for any fixed $n \in \mathbb{N}$, there exist disjoint segments $I_1, \ldots, I_{h_n}$ covering $[0,1]$ and functions $\alpha_1, \ldots, \alpha_{h_n} \in C^\infty(\mathbb{R}^d)$ such that

$$Z^n_t = \alpha_k X_{i_k} \quad \forall t \in I_k, \quad k = 1, \ldots, h_n.$$

Hence

$$\begin{aligned}
P_n &= \overrightarrow{\exp} \int_0^1 Z^n_t \, dt, \\
&= e^{|I_1|\alpha_1 X_{i_1}} \circ \cdots \circ e^{|I_{h_n}|\alpha_{h_n} X_{i_{h_n}}} \\
&= e^{|I_1|\alpha_1 P^{i_1}_* f_{i_1}} \circ \cdots \circ e^{|I_h|\alpha_h P^{i_{h_n}}_* f_{i_{h_n}}} \\
&= (P^{i_1})^{-1} \circ e^{|I_1|(\alpha_1 \circ P^{i_1}) f_{i_1}} \circ P^{i_1} \circ \cdots \\
&\qquad \circ (P^{i_{h_n}})^{-1} \circ e^{|I_h|(\alpha_{h_n} \circ P^{i_{h_n}}) f_{i_{h_n}}} \circ P^{i_{h_n}}\,,
\end{aligned} \tag{1.17}$$

now let $\beta_k = |I_k|(\alpha_k \circ P^{i_k})$, then

$$P_n = (P^{i_1})^{-1} \circ e^{\beta_1 f_{i_1}} \circ P^{i_1} \circ \cdots \circ (P^{i_{h_n}})^{-1} \circ e^{\beta_{h_n} f_{i_{h_n}}} \circ P^{i_{h_n}}\,,$$

and $P_n \in \mathrm{Gr}\mathcal{F}$ by assumption (1.16). $\square$

CHAPTER 2

Controllability of discrete-time dynamics

In this chapter, we deal with the system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^d\,, \tag{2.1}$$

where $\mathcal{F} = \{f_1, \ldots, f_m\}$ is a bracket generating family of vector fields on $\mathbb{R}^d$. We prove, in Theorem 2.6, that given N integer, an orientation preserving diffeomorphisms on $\mathbb{R}^d$, and a neighborhood of it there exist time-varying feedback controls $u_1, \ldots, u_m$ that are polynomial with respect to q and trigonometric polynomial with respect to t such that the exponential, namely the flow at time 1, of system (2.1) belongs to the given neighborhood and the N-jet of the exponential and the N-jet of the given diffeomorphism at the origin coincide.

The structure of the chapter is as follows. Section 2.1 contains some preliminary about jets of functions, vector fields, and diffeomorphisms that we use in the chapter. In Section 2.2 we use classical implicit function theorem applied to the N-jet of the exponential map to prove that the N-jet of a diffeomorphism in $\mathrm{Diff}_0(\mathbb{R}^d)$ sufficiently close to the identity can be presented as the N-jet of an element in $\mathrm{Gr}\mathcal{F}$. Then using Proposition 1.8 we can extend this result to every diffeomorphism in $\mathrm{Diff}_0(\mathbb{R}^d)$. This implies, as showed in Section 2.3, the main result for $f_0 \equiv 0$, and with controls $u_i(t,\cdot)$ that are piecewise constant with respect to t and smooth with respect to q. In Section 2.4 a fixed point argument leads to the proof of Theorem 2.6. The results of this chapter are part of our paper [2].

2.1 Jets

Here we state some simple properties for jets of smooth functions, vector fields, and diffeomorphisms on $\mathbb{R}^d$ that are useful for what follows. For more details we refer to any book of differential geometry or dynamical systems (see for example [5, Section 2.1]). We start with the definition of jet.

Definition 11. An N-jet of a smooth function at the origin 0 of $\mathbb{R}^d$ is defined to be the class of functions whose Taylor expansions at the point 0 coincide up to and including terms of degree N.

Hence, in a fixed coordinate system, an N-jet of a function is given by a polynomial of degree less than or equal to N. Another definition which is clearly independent of the coordinate system and equivalent to the one above is the following.

Definition 12. The N-jet of a smooth function a at the point 0 in $\mathbb{R}^d$ is the class of all functions that are equal to a up to $o(r^N)$ as the distance from the origin r tends to 0. We denote the N-jet at 0 of a as $J_0^N(a)$.

Similarly we can define the jet of a function in an arbitrary point $q_0 \in \mathbb{R}^d$ and r has to be regarded as the distance from q_0. Finally N-jets of vector fields on $\mathbb{R}^d$ or diffeomorphisms are defined similarly.

Given $a, b \in C^\infty(\mathbb{R}^d)$ then we can define the product of their jets via

$$J_0^N(a)J_0^N(b) = J_0^N(ab)$$

where the product is the product of polynomials in $q \in \mathbb{R}^d$ modulo q^{N+1}.

Given P, Q diffeomorphisms of $\mathbb{R}^d$ by chain rule we have

$$J_0^N(P \circ Q) = J_{Q(0)}^N(P) \circ J_0^N(Q),$$

where $\circ$ is the composition of polynomials in $q \in \mathbb{R}^d$ modulo q^{N+1}. Finally, we can also define the inverse of the jet as

$$J_0^N(P)^{-1} = J_{P(0)}^N(P^{-1}).$$

2.2 Get the jet

In this section, given a bracket generating family of vector fields $\mathcal{F}$, we find a diffeomorphism in the group $\mathrm{Gr}_S\mathcal{F}$ whose N-jet is exactly the N-jet of a given diffeomorphism on $\mathbb{R}^d$. The main tool used is the classical implicit function theorem.

Proposition 2.1. *Let $\mathcal{F}$ be a bracket generating family of vector fields on $\mathbb{R}^d$ and $N > 0$ a positive integer.*
For any diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ sufficiently close to the identity there exists $P \in \mathrm{Gr}_S\mathcal{F}$ such that

$$J_0^N(P) = J_0^N(\Phi).$$

Proof. Consider a frame of vector fields $X_1, \dots, X_d$, linearly independent in $0 \in \mathbb{R}^d$. Let $\mathbf{X}$ be the space of polynomials of degree less or equal than N in d variables and let $\mathbf{Y}$ be the space of N-jets at 0 of smooth orientation preserving diffeomorphisms,

i.e. $\mathbf{Y} = J_0^N(\mathrm{Diff}_0(\mathbb{R}^d))$. Note that $\dim \mathbf{X} < \infty$ and $\dim \mathbf{Y} < \infty$.
Consider the map

$$\begin{array}{rccc} F: & \mathbf{X}^d & \longrightarrow & \mathbf{Y} \\ & (u_1, \dots, u_d) & \longmapsto & J_0^N(e^{u_1 X_1} \circ \dots \circ e^{u_d X_d}) \end{array} \tag{2.2}$$

We want to prove that implicit function theorem applies. Let us compute the differential of F at $0 \in \mathbf{X}^d$, we have

$$\begin{aligned} D_0 F(a_1, \dots, a_d) &= \frac{\partial F}{\partial u_1}\Big|_{u_1 = \dots = u_d = 0} a_1 + \dots + \frac{\partial F}{\partial u_d}\Big|_{u_1 = \dots = u_d = 0} a_d \\ &= a_1 J_0^N(X_1) + \dots + a_d J_0^N(X_d). \end{aligned}$$

We claim that $D_0 F : \mathbf{X}^d \to T_{\mathrm{Id}} \mathbf{Y}$ is surjective. Indeed

$$T_{\mathrm{Id}} \mathbf{Y} = T_{\mathrm{Id}} J_0^N(\mathrm{Diff}_0(\mathbb{R}^d)) = J_0^N(T_{\mathrm{Id}} \mathrm{Diff}_0(\mathbb{R}^d)) = J_0^N(\mathrm{Vec}(\mathbb{R}^d)),$$

so for every $V \in J_0^N(\mathrm{Vec}(\mathbb{R}^d))$ there exist $b_1, \dots, b_d \in C^\infty(\mathbb{R}^d)$ such that

$$\begin{aligned} V &= J_0^N(b_1 X_1 + \dots + b_d X_d) \\ &= J_0^N(b_1) J_0^N(X_1) + \ldots + J_0^N(b_d) J_0^N(X_d). \end{aligned}$$

Every element $V \in T_{\mathrm{Id}} \mathbf{Y}$ is image of d polynomials of degree less or equal than N, $a_i = J_0^N(b_i)$. Therefore there exists $\mathcal{O}$ neighborhood of Id in $\mathbf{Y}$ such that F is locally surjective on $\mathcal{O}$. Namely, for every $\psi \in \mathcal{O}$, there exist $u_1, \dots, u_d \in \mathbf{X}$ such that $F(u_1, \dots, u_d) = \psi$. If Φ is sufficiently close to the identity, then $J_0^N(\Phi) \in \mathcal{O}$. Therefore there exist polynomials $v_1, \dots, v_d \in \mathbf{X}$ such that

$$J_0^N(e^{v_1 X_1} \circ \dots \circ e^{v_d X_d}) = J_0^N(\Phi).$$

It remains to prove that $P = e^{v_1 X_1} \circ \dots \circ e^{v_d X_d} \in \mathrm{Gr}_S \mathcal{F}$. According to Orbit Theorem, for $i = 1, \dots, d$, we have $X_i = \mathrm{Ad}\, P^i f_i$, where $f_i \in \mathcal{F}$ and $P^i \in \mathrm{Gr}\mathcal{F}$. Let $P^i = e^{t_1^i f_1^i} \circ e^{t_2^i f_2^i} \circ \dots \circ e^{t_{s_i}^i f_{s_i}^i}$ with $f_j^i \in \mathcal{F}$. Therefore

$$\begin{aligned} P &= e^{v_1 \mathrm{Ad}\, P^1 f_1} \circ \dots \circ e^{v_d \mathrm{Ad}\, P^d f_d} \\ &= P^1 \circ e^{(P^1)^{-1}(v_1) f_1} \circ \left(P^1\right)^{-1} \circ \dots \circ P^d \circ e^{\left(P^d\right)^{-1}(v_d) f_d} \circ \left(P^d\right)^{-1} \\ &= \underbrace{e^{t_1^1 f_1^1} \circ \dots \circ e^{t_{s_1}^1 f_{s_1}^1}}_{P^1} \circ e^{(P^1)^{-1}(v_1) f_1} \circ \underbrace{e^{-t_{s_1}^1 f_{s_1}^1} \circ \dots \circ e^{-t_1^1 f_1^1}}_{(P^1)^{-1}} \circ \dots \\ &\quad \circ \underbrace{e^{t_1^d f_1^d} \circ \dots \circ e^{t_{s_d}^d f_{s_d}^d}}_{P^d} \circ e^{\left(P^d\right)^{-1}(v_d) f_d} \circ \underbrace{e^{-t_{s_d}^d f_{s_d}^d} \circ \dots \circ e^{-t_1^d f_1^d}}_{\left(P^d\right)^{-1}} \\ &= e^{w_1 g_1} \circ \dots \circ e^{w_\ell g_\ell}, \end{aligned} \tag{2.3}$$

with $g_1, \dots, g_\ell \in \mathcal{F}$ and $\ell = d + 2(s_1 + \dots + s_d)$. Therefore $P \in \mathrm{Gr}_S \mathcal{F}$ and the proposition follows. □

Now consider any diffeomorphism $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$. By Proposition 1.8 there exists a sequence $\{P_n\}_n \subset \mathrm{Gr}_S\mathcal{F}$ that tends to Φ. So, for n sufficiently large, last Proposition applies to $P_n^{-1} \circ \Phi$ and we have the following result.

Corollary 2.2. *Let $\mathcal{F} \subseteq \mathrm{Vec}\,\mathbb{R}^d$ be a bracket generating family of vector fields and $N > 0$ a positive integer. For every $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$ there exists $P \in \mathrm{Gr}_S\mathcal{F}$ such that*

$$J_0^N(P) = J_0^N(\Phi).$$

2.3 The homogeneous case

The purpose of this section is to link the results of the last section with Proposition 1.8 in order to find an element in the group $\mathrm{Gr}_S\mathcal{F}$ with the same N-jet of a given diffeomorphism and also close to it in the C^∞-topology.

Proposition 2.3. *Let $\mathcal{F} \subseteq \mathrm{Vec}\mathbb{R}^d$ be a bracket generating family of vector fields. Let N and k be positive integers, $\varepsilon > 0$, and B ball of $\mathbb{R}^d$. For any $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exists $P \in \mathrm{Gr}_S\mathcal{F}$ such that*

$$J_0^N(P) = J_0^N(\Phi) \quad \textit{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof. We can suppose that $J_0^N(\Phi) = \mathrm{Id}$. Indeed, by Corollary 2.2, there exists $Q \in \mathrm{Gr}_S\mathcal{F}$ such that $J_0^N(Q) = J_0^N(\Phi)$. Then we consider, instead of Φ, the diffeomorphism $\Psi = \Phi \circ Q^{-1}$ which has trivial jet.

The idea of the proof is the same of Proposition 1.8. Since $J_0^N(\Phi) = \mathrm{Id}$, then Φ can be written as

$$\Phi(x) = x + g(x),$$

with $J_0^N(g) = 0$. Consider the one parameter family of diffeomorphisms with trivial jet

$$\Phi_t(x) = x + tg(x)\,.$$

This is a path in $\mathrm{Diff}(\mathbb{R}^d)$ from $\Phi_0 = \mathrm{Id}$ to $\Phi_1 = \Phi$. Let V_t a nonautonomous vector field such that

$$\Phi_t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau.$$

Let $X_1, \ldots, X_d$ be a frame of vector fields linearly independent at 0 such that $X_i = \mathrm{Ad}\, P^i f_i$ with $P^i \in \mathrm{Gr}\mathcal{F}$ and $f_i \in \mathcal{F}$. Therefore

$$V_t = \sum_{i=1}^{d} a_i(t,\cdot) X_i,$$

with $a_i(t,\cdot) \in C^\infty(\mathbb{R}^d)$ for any $t \in [0,1]$. Note that, since $J_0^N(\Phi_t) = \mathrm{Id}$ and the vector fields X_i are linearly independent, then $J_0^N(a_i(t,\cdot)) = 0$ for any $t \in [0,1]$.

Now let $\mathcal{A}$ be the closed subspace of $C^\infty(\mathbb{R}^d)$ of smooth functions α such that $J_0^N(\alpha) = 0$. By Proposition 1.7 there exists a sequence of nonautonomous vector field $Z_t^n \in \{\alpha X_i \,|\, \alpha \in \mathcal{A}, i = 1\ldots, d\}$, piecewise constant in t, such that

$$\overrightarrow{\exp} \int_0^t Z_\tau^n \, d\tau \to \Phi_t, \quad \text{as } n \to \infty,$$

in the C^∞-topology and uniformly with respect to $t \in [0,1]$.
So, if $P_n = \overrightarrow{\exp} \int_0^1 Z_\tau^n \, d\tau$, then

$$P_n \to \Phi, \quad \text{as } n \to \infty,$$

in the standard topology. Now, for any n, we have that $P_n \in \mathrm{Gr}_S\mathcal{F}$ for the chain of equalities (1.17). Moreover P_n has trivial jet. Indeed, since the sequence Z_t^n is piecewise constant, then there exist intervals $I_1, \ldots, I_h$ such that

$$Z_t^n = \alpha_i X_{j_i} \quad \text{for any } t \in I_i,$$

with $j_i \in \{1, \ldots, d\}$. So

$$\begin{aligned} J_0^N(P_n) &= J_0^N\left(\overrightarrow{\exp} \int_0^1 Z_t^n \, dt\right) \\ &= J_0^N\left(e^{|I_1|\alpha_1 X_{j_1}}\right) \circ \cdots \circ J_0^N\left(e^{|I_h|\alpha_h X_{j_h}}\right) \\ &= e^{|I_1| J_0^N(\alpha_1) J_0^N(X_{j_1})} \circ \cdots \circ e^{|I_h| J_0^N(\alpha_h) J_0^N(X_{j_h})} \\ &= \mathrm{Id}\,, \end{aligned}$$

and the result is proved. □

2.4 Adding drift and smoothing

In this last section we prove the main result of this chapter using Proposition 2.3 and a fixed point argument. We start giving an equivalent formulation of Proposition 2.3 in terms of flows of the homogeneous (with respect to control) system:

$$\dot{q} = \sum_{i=1}^m u_i(t,q) f_i(q), \qquad q \in \mathbb{R}^d\,. \tag{2.4}$$

Suppose that $\mathcal{F} = \{f_1, \ldots, f_m\}$ is a bracket generating family of vector field on $\mathbb{R}^d$. By Proposition 2.1 there exist smooth functions $a_1, \ldots, a_k$ such that

$$J_0^N(\Phi) = J_0^N\left(e^{a_1 f_{i_1}} \circ \cdots \circ e^{a_k f_{i_k}}\right)\,, \tag{2.5}$$

with $i_j \in \{1,\dots,m\}$. Now there exist m functions $u_1(t,q),\dots,u_m(t,q)$ piecewise constant in t such that

$$J_0^N(\Phi) = J_0^N\left(\overrightarrow{\exp}\int_0^1 \sum_{i=1}^m u_i(t,\cdot) f_i\,dt\right). \tag{2.6}$$

We then proved the following Lemma.

Lemma 2.4. *Let $\{f_1, f_2, \dots, f_m\}$ be a bracket generating family of vector fields on $\mathbb{R}^d$. Consider the control system*

$$\dot q = \sum_{i=1}^m u_i(t,q) f_i(q)\,, \qquad q \in \mathbb{R}^d\,, \tag{2.7}$$

with controls u_i piecewise constant with respect to $t \in [0,1]$ and smooth with respect to $q \in \mathbb{R}^d$, for every $i = 1,\dots,m$.
Let N and k be positive integers, $\varepsilon > 0$, and B ball in $\mathbb{R}^d$. For any $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t,q),\dots,u_m(t,q)$ such that, if P is the flow at time 1 of system (2.7), then

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

It remains to prove last result adding a drift f_0 to system (2.7). Moreover we want the controls to be of a certain regularity. Both these results can be achieved with a fixed point argument. Indeed, let $\mathbf{U}$ the space of m-uples of controls $u(t,q)$ piecewise constant in t and smooth with respect to q. Consider the map

$$\begin{array}{cccc} \tilde F: & \mathbf{U} & \longrightarrow & J_0^N(\mathrm{Diff}_0(\mathbb{R}^d)) \\ & (u_1,\dots,u_m) & \longmapsto & J_0^N\left(\overrightarrow{\exp}\int_0^1 \sum_{i=1}^m u_i(t,\cdot) X_i\,dt\right). \end{array} \tag{2.8}$$

This map is continuous and, by last Lemma, it is also surjective. Moreover $\tilde F$ has a continuous right inverse. Indeed there is a smooth correspondence between the time-varying feedback controls $u_1,\dots,u_m$ and the functions $a_1,\dots,a_k$ in (2.5). By implicit function theorem applied to the map F in (2.2), we have that the right inverse of F is continuous and so is the right inverse of $\tilde F$.
In the next Lemma we prove, using a fixed point argument, that every small perturbation of a continuous surjective map with continuous right inverse and with finite dimensional target space is surjective too.

Lemma 2.5. *Let X be a topological space, $\varepsilon > 0$, and let $F : X \to \mathbb{R}^n$ be continuous and surjective with continuous right inverse. If $G : X \to \mathbb{R}^n$ is continuous and such that $\sup_{x\in K} |F(x) - G(x)| < \varepsilon$ for any $K \subseteq X$ compact, then G is surjective.*

Proof. Let F^{-1} be the right inverse of F and define, for every $\bar{y}$ in $\mathbb{R}^n$, the map $\chi_{\bar{y}}(y) = y - G \circ F^{-1}(y) + \bar{y}$. Let $\delta = \varepsilon + \|\bar{y}\|$, then for every $y \in B_\delta = B_\delta(0)$ we have

$$\begin{aligned}
\|\chi_{\bar{y}}(y)\| &\leq \|y - G \circ F^{-1}(y)\| + \|\bar{y}\| \\
&\leq \sup_{y \in B_\delta} \|y - G \circ F^{-1}(y)\| + \|\bar{y}\| \\
&\leq \sup_{x \in F^{-1}(B_\delta)} \|F(x) - G(x)\| + \|\bar{y}\| \\
&< \varepsilon + \|\bar{y}\| \\
&= \delta.
\end{aligned}$$

So $\chi_{\bar{y}}(B_\delta) \subseteq B_\delta$ and, since the map $\chi_{\bar{y}}$ is continuous, by Brouwer Fixed Point Theorem, there exists $\tilde{y} \in B_\delta$ such that

$$\chi_{\bar{y}}(\tilde{y}) = \tilde{y},$$

namely

$$G \circ F^{-1}(\tilde{y}) = \bar{y}.$$

We proved that, for every $y \in \mathbb{R}^n$, there exists, $x \in X$ such that $y = G(x)$. □

The main result of this chapter can now be proved.

Theorem 2.6. *Let $\{f_1, f_2, \ldots, f_m\}$ be a bracket generating family of vector fields on $\mathbb{R}^d$. Consider the control system*

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^d, \tag{2.9}$$

with controls u_i such that:

(i) u_i is polynomial with respect to $q \in \mathbb{R}^d$;

(ii) u_i is a trigonometric polynomial with respect to $t \in [0,1]$;

for every $i = 1, \ldots, m$.
Fix positive integers N and k, $\varepsilon > 0$, and B ball of $\mathbb{R}^d$. For any $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t,q), \ldots, u_m(t,q)$ such that, if P is the flow at time 1 of system (2.9), then

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof. Proof splits into three steps. First, we prove that it is not restrictive to consider controls that are polynomials with respect to $q \in \mathbb{R}^d$, then we add the drift to the system, and finally we find controls that are trigonometric polynomials with respect to t by smoothing the time dependence of the piecewise constant controls.

Let us start with the first step and note that, as a consequence of the density of polynomials in the space of smooth functions on a bounded set and by Lemma 2.5, we can assume that $u_i(t,q)$ is a polynomial in q for every $t \in [0,1]$ and for every $i = 1, \dots, m$.

Now consider the family of continuous maps

$$F_\varrho : (u_1, \dots, u_m) \mapsto J_0^N \left(\overrightarrow{\exp} \int_0^{1/\varrho} \varrho f_0 + \sum_{i=1}^m u_i(t,\cdot) X_i \, dt \right).$$

We claim that, if there exists $\varrho > 0$ such that F_ϱ is surjective then so is F_ϱ for $\varrho = 1$. Indeed

$$F_\varrho(u_1(t,\cdot), \dots, u_m(t,\cdot)) = F_1 \left(\frac{u_1(t/\varrho,\cdot)}{\varrho}, \dots, \frac{u_m(t/\varrho,\cdot)}{\varrho} \right),$$

similarly the map $\tilde{F}_\varrho(u_1, \dots, u_m) = J_0^N \left(\overrightarrow{\exp} \int_0^{1/\varrho} \sum_{i=1}^m u_i(t,\cdot) X_i \, dt \right)$ is surjective for every $\varrho > 0$ since it is equal to the map $\tilde{F}$ defined in (2.8) up to rescalings of the time dependence of the controls u_i, namely

$$\tilde{F}_\varrho(u_1(t,\cdot), \dots, u_m(t,\cdot)) = \tilde{F} \left(\frac{u_1(t/\varrho,\cdot)}{\varrho}, \dots, \frac{u_m(t/\varrho,\cdot)}{\varrho} \right),$$

For small $\varrho > 0$, F_ϱ is a small perturbation of $\tilde{F}_\varrho$, thus Lemma 2.5 applies and F_1 is surjective.

Finally, any control $u(t,q)$ piecewise constant in t and polynomial in q, can be written

$$u(t,q) = \sum_{|\alpha|=0}^N a_\alpha(t) q^\alpha,$$

with α multi-index and $a_\alpha(t)$ piecewise constant. For every α, the function a_α admits a Fourier expansion of the form

$$a_\alpha(t) = \sum_{j=0}^\infty \eta_\alpha^j \cos(2\pi j t) + \xi_\alpha^j \sin(2\pi j t).$$

Consider the trigonometric polynomial

$$a_\alpha^n(t) = \sum_{j=0}^n \eta_\alpha^j \cos(2\pi j t) + \xi_\alpha^j \sin(2\pi j t),$$

then $a_\alpha^n(t) \to a_\alpha(t)$ as $n \to \infty$ in $L^1[0,1]$. So let

$$u^n(t,q) = \sum_{|\alpha|=0}^N a_\alpha^n(t) q^\alpha,$$

then

$$u^n(t,q) \to u(t,q), \quad \text{as } n \to \infty, \tag{2.10}$$

and the convergence is uniform with all derivatives on compact sets of $\mathbb{R}^d$ and in $L^1[0,1]$ with respect to t.
Let G_n be the family of continuous maps

$$\begin{array}{rccc} G_n : & \mathbf{U} & \longrightarrow & \mathbf{Y} \\ & (u_1,\ldots,u_m) & \longmapsto & J_0^N \left(\overrightarrow{\exp} \int_0^1 f_0 + \sum_{i=1}^m u_i^n(t,\cdot) X_i \, dt \right). \end{array}$$

By the convergence in (2.10), $G_n \to F_1$ as $n \to \infty$ for every $(u_1,\ldots,u_m) \in \mathbf{U}$, then there exists n_0 integer for which Lemma 2.5 applies. Therefore the map G_{n_0} is surjective and Theorem follows. □

Remark 2. Clearly the statement of Theorem 2.6 holds also if we consider the jet at a point $q \in \mathbb{R}^d$. Moreover it is possible to fix a finite number of points in $\mathbb{R}^d$, say $q_1,\ldots,q_\ell$, and find an admissible diffeomorphism, arbitrary close to a given one, that realize its N-th jet at all the points $q_1,\ldots,q_\ell$ at the same time.

CHAPTER 3

Controllability on the group of diffeomorphisms

In this chapter we prove the main result of the thesis, Theorem 3.1, also stated in the Introduction. In particular, in Section 3.1 we state the result and we present some immediate corollary. In Section 3.2 we explain the strategy adopted to prove the result and a description of why classical implicit function theorem does not apply is provided. Then we start the proof of Theorem 3.1 showing, in Section 3.3, an auxiliary result concerning local diffeomorphisms in $\mathbb{R}^d$. Namely, given d vector fields over $\mathbb{R}^d$, $X_1, \ldots, X_d$, linearly independent at the origin, we find a closed neighborhood V of the origin in $\mathbb{R}^d$ such that the image of the map

$$F : (a_1, \ldots, a_d) \mapsto e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d}\big|_V$$

from $C_0^\infty(\mathbb{R}^d)^d$ to $C_0^\infty(V)^d$ has nonempty interior. In Section 3.4 we show how to reduce the proof of Theorem 3.1 to the mentioned auxiliary fact using a geometric idea that goes back to the Orbit Theorem of Sussmann. The results of this chapter are contained in our paper [1].

3.1 Statement and corollaries

Throughout this chapter, if not otherwise specified, M denotes a compact connected smooth manifold.

The main result of this work is as follows

Theorem 3.1. *Let $\mathcal{F} \subset \operatorname{Vec} M$ be a family of smooth vector fields such that $\operatorname{Gr}\mathcal{F}$ acts transitively on M. Then there exist a neighborhood $\mathcal{O}$ of the identity in $\operatorname{Diff}_0(M)$ and a positive integer μ such that every $P \in \mathcal{O}$ can be presented in the form*

$$P = e^{a_1 f_1} \circ \cdots \circ e^{a_\mu f_\mu} ,$$

for some $f_1, \ldots, f_\mu \in \mathcal{F}$ and $a_1, \ldots, a_\mu \in C^\infty(M)$.

In particular, if $\mathcal{F}$ is a bracket generating family of vector fields then by Chow–Rashevsky Theorem $\operatorname{Gr}\mathcal{F}$ acts transitively on M. By the the density of $\operatorname{Gr}_S\mathcal{F}$ in $\operatorname{Diff}_0(M)$, Proposition 1.8, then any diffeomorphism in $\operatorname{Diff}_0(M)$ can be presented as composition of exponentials of vector fields in $\mathcal{F}$ rescaled by smooth functions. In fact, the theorem states a stronger result, namely that every diffeomorphism sufficiently close to the identity can be presented as the composition of a number of exponentials μ that depends only on the distribution $\mathcal{F}$.

Remark 3. An open problem related to Theorem 3.1 could be to study what is the number μ of exponentials needed to represent diffeomorphisms sufficiently close to the identity. This number depends only on the distribution and the interesting problem is to estimate μ in terms of simple discrete invariants like the growth vector of the distribution and the Lusternik–Schnirelmann category (see [15]) of the manifold.

Here we give some simple corollaries to Theorem 3.1. A first direct consequence is the following.

Corollary 3.2. *Let $\mathcal{F} \subset \mathrm{Vec}M$, if $\mathrm{Gr}\mathcal{F}$ acts transitively on M, then $\mathrm{Gr}_S\mathcal{F} = \mathrm{Diff}_0(M)$.*

Let us reformulate last corollary in terms of control systems. Consider the driftless control system on M

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in M, \tag{3.1}$$

where $\{f_1, \dots, f_m\}$ is a bracket generating family of vector fields and $u_1, \dots, u_m$ are time varying feedback controls, piecewise constant in $t \in [0,1]$ for every q. Corollary 3.2 states that for every $P \in \mathrm{Diff}_0(M)$ there exist time-varying feedback controls $u_1, \dots, u_m$, such that $q(1) = P(q(0))$ for any solution $q(\cdot)$ of system (3.1); in other words,

$$P = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt. \tag{3.2}$$

In fact, to any representation of P as finite product of exponentials

$$P = e^{a_1 f_{j_1}} \circ \cdots \circ e^{a_\ell f_{i_\ell}} ,$$

it corresponds a m-uple of time–varying feedback controls that are piecewise constant with respect to time, such that P is the flow, at time 1, of system (3.1).
Now, consider a manifold M, eventually noncompact, given $P \in \mathrm{Diff}_0(M)$, on every compact subset $K \subset M$, P has a representation of the form (3.2), with $u_i(t,\cdot)$ piecewise constant with respect to t. Then consider a sequence of compacta K_n such that

$$K_n \subset K_{n+1}\,, \text{ and } \bigcup_{n=1}^{\infty} K_n = M\,,$$

and let μ_n be a sequence of cut–off functions with compact support K_n, for every $n \geq 1$, such that $\mu_n \to 1$ as $n \to \infty$. Then, for every n, there exist controls $u_i^n(t, \cdot)$ piecewise constant in t, such that

$$\mu_n P = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i^n(t, \cdot) f_i \, dt \, .$$

Moreover, we can suppose that the controls $u^n = (u_1^n, \ldots, u_m^n)$ are such that

$$u^n\big|_{[0,1]\times K_{n-1}} = u^{n+1}\big|_{[0,1]\times K_{n-1}} \, .$$

Indeed, on every compact set K_n we can choose an exponential representation of a diffeomorphism. Then, at every step the control u^{n+1} adds informations about the representation on the set $K_{n+1} \setminus K_n$. Therefore we can suppose that u^{n+1} coincides, at least on K_{n-1}, with u^n. Then we can take the limit as $n \to \infty$ and extend the main result to a manifold eventually noncompact. Therefore we have the following corollary.

Corollary 3.3. *Let M be a connected manifold, if a system on M is bracket generating and driftless then*

$$\mathbf{A}_t = \mathrm{Diff}_0(M)$$

for every $t > 0$.

Finally, next corollary is stated from a geometric viewpoint, in terms of completely nonholonomic vector ditributions.

Corollary 3.4. *Let $\Delta \subset TM$ be a completely nonholonomic vector distribution. Then every diffeomorphism of M that is isotopic to the identity can be written as $e^{f_1} \circ \cdots \circ e^{f_k}$, where $f_1, \ldots, f_k$ are sections of Δ.*

3.2 Proof strategy

A natural strategy to prove Theorem 3.1 could be to use the underlying idea of the proof of Proposition 2.1. The core of that proof consists in the fact that the map F defined in (2.2) is locally onto. Surjectivity of the map F is a consequence of implicit function theorem, so our problem is to determine whether, given $U \subset \mathbb{R}^d$ with $0 \in U$ and $X_1, \ldots, X_d$ vector fields linearly independent 0, the exponential map

$$\begin{array}{rccc} F: & C^\infty(U)^d & \to & \mathrm{Diff}_0(U) \\ & (a_1, \ldots, a_d) & \mapsto & e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d}\big|_U . \end{array} \tag{3.3}$$

is locally onto. Unfortunately the classical inverse function theorem does not apply for the exponential map neither in dimension $d = 1$ on the circle S^1. Indeed the exponential of the 0 vector field is the identity of the group of diffeomorphisms,

$e^0 = \mathrm{Id}$, and the derivative of the exponential map at the vector field 0 is the identification of the vector fields with the tangent space of the diffeomorphisms at the identity. Therefore, we have a smooth map of a vector space, $\mathrm{Vec}\, M$, to a manifold, $\mathrm{Diff}(M)$, whose derivative at 0 is the identity map from the vector space to the tangent space of the manifold. If inverse function theorem applies then the exponential map is locally invertible. Nevertheless the exponential map fails to be locally surjective in a neighborhood of the identical diffeomorphism as the following remark due to Hamilton [19] shows.

Remark 4. Consider the circle S^1. We write any diffeomorphism on S^1 as a smooth 2π–periodic function of the variable x. So x is a parametrization of the circle modulo 2π. Vector fields over S^1 are of the form $v(x)\partial_x := v(x)\frac{\partial}{\partial x}$ with v smooth and 2π-periodic. If $v(x) = \gamma$ is constant, then the exponential of $\gamma\partial_x$ is the rotation of angle γ. We start with the following rather surprising fact about exponentials of vector fields in S^1.

Proposition 3.5. *If a diffeomorphism of the circle without fixed points is the exponential of a vector field, then it is conjugate to a rotation.*

Proof. Let V be a vector field of the circle and consider the exponential of V. If V has a zero then e^V has a fixed point. Therefore in the coordinate x, $V = v(x)\partial_x$ with $v(x) \neq 0$ for every $x \in [0, 2\pi]$. Now let

$$\gamma = 2\pi / \int_0^{2\pi} \frac{1}{v(x)}\, dx \,,$$

and consider the change of coordinates

$$x \mapsto \theta = \gamma \int_0^x \frac{1}{v(s)}\, ds \,.$$

Then θ is a new parametrization of the circle modulo 2π and

$$V = v(x)\partial_x = \gamma\partial_\theta.$$

Therefore, in the new coordinate θ, the exponential of V is the rotation of angle γ. □

Now let $f \in C^\infty_{2\pi}$ such that $f = e^{v\partial_x}$ and suppose that f is conjugate to rotation, namely that there exist a change of coordinates Φ on S^1 and $\gamma \in [0, 2\pi)$ such that $\Phi \circ f \circ \Phi^{-1} = e^{\gamma\partial_x}$. If such an f fixes one point then it must fix them all. Indeed, a rotation that fixes one point is the identity and if there exist $\bar{x}$ such that $f(\bar{x}) = \bar{x}$ then we have that the rotation of angle γ fixes the point $\Phi(\bar{x})$, which implies $\gamma = 0$ and $f = \mathrm{Id}$. Clearly the same statement is true for any power $f^k = f \circ \cdots \circ f$. Therefore, in order to find a diffeomorphism that is not a rotation up to change of coordinates, we have to look for a diffeomorphism f such that $f^k(x) = x$ for some $x \in S^1$ and $f^k(y) \neq y$ for some other $y \in S^1$ and for some integer k. Consider the rotation of angle $2\pi/k$, say $h = e^{\frac{2\pi}{k}\partial_x}$. Then $h^k(x) = x \pmod{2\pi}$ for every $x \in S^1$. Now consider a smooth bump function b such that $\operatorname{supp} b \subset (0, 2\pi/k)$ and such that $b(\pi/k) > \pi/k$. Let $f = h + b$, then $f(0) = 0$ but $f(\pi/k) > 3\pi/k$. Iterating f we have $f^k(0) = 0$ and $f^k(\pi/k) \neq \pi/k \pmod{2\pi}$. Then this diffeomorphism cannot be the exponential of a vector field. Taking, if necessary, k large and b small we can suppose f as close to the identity as we want in the C^∞ topology.

Therefore we have to adopt a different strategy. The idea used in Section 3.3 allows us to prove surjectivity of the map (3.3) directly without studying its differential. Let us consider again the 1-dimensional case. Consider the problem locally and let U a neighborhood of the origin of $\mathbb{R}$, consider a linear diffeomorphism on U, say $x \mapsto \alpha x\big|_U$. We know by remark 4 that near the identity the exponential is not onto so we have to assume $\alpha > 1$. Then $\beta = \log \alpha$ is well defined and positive. Now it is easy to see that the vector field $V(x) = \beta x \frac{\partial}{\partial x}$ is the required one since $e^V : x \mapsto \alpha x$. Since we have an exponential representation for linear diffeomorphisms the idea for the proof of Proposition 3.6 reduces in finding a time dependent change of coordinates that linearizes a given diffeomorphism. It turns out that such a change of coordinates can be found starting from a first order linear PDE, easy solvable with the method of characteristics. Then Lemma 3.7 allows us to consider the problem in dimension 1. Finally, in Section 3.4, we show how to reduce the proof of Theorem 3.1 to the proof of the surjectivity of the map (3.3) using the Orbit Theorem of Sussmann, Theorem 1.1. and in particular Remark 1.

Remark 5. Note that proof strategy is rather effective and, in principle, could be implemented as a numerical algorithm, that find the controls, at least approximately, from a given diffeomorphism.

3.3 A direct proof of surjectivity

Proposition 3.6. *Let $X_1, \dots, X_d \in \operatorname{Vec} \mathbb{R}^d$ be such that*

$$\operatorname{span}\{X_1(0), \dots, X_d(0)\} = \mathbb{R}^d .$$

Then there exist a compact neighborhood V of the origin in $\mathbb{R}^d$ and a open subset $\mathcal{V}$ of $C_0^\infty(V)^d$ such that every $F \in \mathcal{V}$ can be written as

$$F = e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d}\big|_V ,$$

for some $a_1, \dots, a_d \in C_0^\infty(\mathbb{R}^d)$.

In order to prove this result we need the following Lemma.

Lemma 3.7. *Let $X_1, \dots, X_d \in \operatorname{Vec} \mathbb{R}^d$ be such that*

$$\operatorname{span}\{X_1(0), \dots, X_d(0)\} = \mathbb{R}^d ,$$

and let $\mathcal{U}_0$ be a neighborhood of the identity in $C_0^\infty(\mathbb{R}^d)^d$. Then, there exist a neighborhood V of the origin in $\mathbb{R}^d$ and a neighborhood $\mathcal{U}$ of the identity in $C_0^\infty(V)^d$ such that for every $F \in \mathcal{U}$ there exist $\varphi_1, \dots, \varphi_d \in \mathcal{U}_0$ such that

$$F = \varphi_1 \circ \cdots \circ \varphi_d\big|_V ,$$

where φ_k preserves the 1-foliation generated by the trajectories of the equation $\dot{q} = X_k(q)$ for $k = 1, \dots, d$.

Proof. Since $X_1, \dots, X_d$ are linearly independent at 0 then there exists a neighborhood of the origin $V \subset \mathbb{R}^d$ such that

$$\operatorname{span}\{X_1(q), \dots, X_d(q)\} = \mathbb{R}^d , \qquad \text{for every } q \in \bar{V} .$$

Now, there exists a ball $B \subset \mathbb{R}^d$ containing $0 \in \mathbb{R}^d$ such that, for every $q \in V$, the map

$$(t_1, \dots, t_d) \mapsto q \circ e^{t_1 X_1} \circ \cdots \circ e^{t_d X_d} , \tag{3.4}$$

is a local diffeomorphism from B to a neighborhood of q. Let

$$\mathcal{U}_\varepsilon = \{F \in C_0^\infty(V)^d \,:\, \|F - \operatorname{Id}\|_{C^1} < \varepsilon\} ,$$

where ε is to be chosen later. If ε is sufficiently small, then, for every $F \in \mathcal{U}_\varepsilon$, $F(q)$ belongs to the image of map (3.4). Therefore, given every $F \in \mathcal{U}_\varepsilon$, it is possible to associate with every $q \in V$ a d-uple of real numbers $(t_1(q), \dots, t_d(q)) \in B$ such that

$$F(q) = q \circ e^{t_1(q) X_1} \circ \cdots \circ e^{t_d(q) X_d} .$$

We claim that there exists $\eta(\varepsilon)$ such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\|t_i\|_{C^1} < \eta(\varepsilon)$ for every $i = 1, \dots, d$ and for $F \in \mathcal{U}_\varepsilon$. Indeed, $\|F - \operatorname{Id}\|_{C^0} < \varepsilon$ implies that $\|t_i\|_{C^0} < c\varepsilon$, for $i = 1, \dots, d$ and for some constant c. Moreover, if $q \in V$, for every $\xi \in \mathbb{R}^d$ we have

$$D_q F \xi = \left(e^{t_1(q) X_1} \circ \cdots \circ e^{t_d(q) X_d}\right)_* \xi + \sum_{i=1}^{d} e^{t_1(q) X_1} \circ \cdots \circ \frac{dt_i}{dq} \cdot \xi X_i \circ \cdots \circ e^{t_d(q) X_d} .$$

Therefore $\|D_q F \xi - \xi\|_{C^0} < \varepsilon$ implies $\|t_i\|_{C^1} < \eta(\varepsilon)$, where $\eta \to 0$ as $\varepsilon \to 0$.

Now consider, for every $k = 1, \dots, d$, the map

$$\Phi_k(q) = q \circ e^{t_1(q) X_1} \circ \cdots \circ e^{t_k(q) X_k} .$$

Note that $\Phi_0 = I$ and $\Phi_d = F$. For every k, Φ_k is a smooth diffeomorphism being smooth and invertible by the implicit function theorem. Indeed, for every $q \in V$ the differential of Φ_k at q is

$$D_q\Phi_k\xi = \left(e^{t_1(q)X_1} \circ \cdots \circ e^{t_k(q)X_k}\right)_* \xi + \sum_{i=1}^{k} e^{t_1(q)X_1} \circ \cdots \circ \frac{dt_i}{dq}\xi X_i \circ \cdots \circ e^{t_k(q)X_k}\,.$$

Denote by $T(\xi) = D_q\Phi_k\xi - \xi$. For ε sufficiently small $\|T\|_0 < 1$ is a contraction and, therefore, $D_q\Phi_k = \mathrm{Id} + T$ is invertible.

Finally call $\mathcal{U} = \mathcal{U}_\varepsilon$ and define for every $k = 1, \ldots, d$, the smooth maps

$$\varphi_k(q) = q \circ e^{t_k(\Phi_{k-1}^{-1}(q))X_k}\,,$$

then the statement follows. □

Thanks to last Lemma our problem is to find an appropriate exponential representation of every of the functions φ_k. Recall that the idea of the proof of Proposition 3.6 is that a linear diffeomorphism is the exponential of a linear vector field. So, our goal is to find a change of coordinates that linearizes φ_k along trajectories of the equation $\dot{q} = X_k(q)$.

Proof of Proposition 3.6. Let V, $\mathcal{U}$, and $\mathcal{U}_0$ as in Lemma 3.7. Denote by $\mathcal{X}_k$ the set of all $\varphi \in \mathcal{U}_0$ such that φ preserves the 1-foliation generated by the equation $\dot{q} = X_k(q)$. Every $F \in \mathcal{U}$ can be written as $F = \varphi_1 \circ \cdots \circ \varphi_d\big|_V$. Now consider the open subset of $C_0^\infty(V)^d$

$$\begin{aligned}\mathcal{V} \subseteq \{F \in \mathcal{U} : F = \varphi_1 \circ \cdots \circ \varphi_d\big|_V,\ \varphi_k \in \mathcal{X}_k,\\ (D_q\varphi_k)X_k(q) \neq X_k(q) \text{ for } q \in \varphi_k^{-1}(0),\ k = 1, \ldots, d\}\,.\end{aligned}$$

Since every $F \in \mathcal{U}$ is close to the identity, then so is φ_k for every k. Moreover, $\varphi_k(0) = 0$ and X_k transversal to the hypersurface $\varphi_k^{-1}(0)$ at any point. Therefore we may rectify the field X_k in a neighborhood of the origin in such a way that, in new coordinates, $\varphi_k(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_d) = 0$ and $X_k = \frac{\partial}{\partial x_k}$. Set $x := x_k$ and $y := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d)$.
Since the following argument does not depend on $k = 1, \ldots, d$ the subscript k is omitted.

Let $\alpha(y) = \log(\frac{\partial}{\partial x}\varphi(0, y))$. Note that by the definition of $\mathcal{V}$ we have $\alpha(y) \neq 0$ for every y. In what follows we treat y as a $(d-1)$-dimensional parameter and, for the sake of readability, we omit it. We will show, step by step, that the argument holds for every value of the parameter y and all maps and vector fields under consideration depend smoothly on y. Consider the homotopy from φ to the identity

$$\varphi_t(x) = e^{\alpha(t-1)}\varphi(tx)/t, \quad t \in [0, 1].$$

There exists a nonautonomous vector field $a(t,x)\frac{\partial}{\partial x}$ such that

$$\varphi_t = \overrightarrow{\exp} \int_0^t a(\tau,\cdot)\frac{\partial}{\partial x}\, d\tau.$$

It is easy to see that $\frac{\partial a}{\partial x}(t,0)=\alpha$. Indeed

$$\begin{aligned} a(t,\varphi_t(x)) &= \frac{d}{dt}\varphi_t(x) \\ &= \alpha e^{\alpha(t-1)}\frac{\varphi(tx)}{t} + e^{\alpha(t-1)}\frac{\varphi'(tx)x}{t} - e^{\alpha(t-1)}\frac{\varphi(tx)}{t^2} \\ &= \alpha\varphi_t(x) + \frac{\varphi_t'(x)x}{t} - \frac{\varphi_t(x)}{t}\,, \end{aligned}$$

so, differentiating with respect to x and evaluating at $x=0$, we get

$$\frac{\partial a}{\partial x}(t,0)\varphi_t'(0) = \alpha\varphi_t'(0)\,,$$

where $\varphi_t'(0) = e^{\alpha t} \neq 0$. Therefore let $a(t,x) = \alpha x + b(t,x)x$ with $b(t,0)=0$. We want to find a time-dependent change of coordinates $\psi(t,x)$ that linearizes the flow generated by $a(t,x)$. Namely if $x(t)$ is a solution of $\dot{x} = a(t,x)$ and $z(t) = \psi(t,x(t))$ then we want $\dot{z}(t) = \alpha z(t)$. We can suppose $\psi(t,0)=0$ and write $\psi(t,x) = xu(t,x)$, where $u(0,x)=1$. On one hand we have

$$\begin{aligned} \frac{d}{dt}z &= \frac{d}{dt}(xu(t,x)) \\ &= \dot{x}u(t,x) + x\dot{x}\frac{\partial u}{\partial x}(t,x) + x\frac{\partial u}{\partial t}(t,x) \\ &= a(t,x)u(t,x) + xa(t,x)\frac{\partial u}{\partial x}(t,x) + x\frac{\partial u}{\partial t}(t,x)\,, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \frac{d}{dt}z &= \alpha z \\ &= \alpha x u(t,x)\,. \end{aligned}$$

Therefore, we can find u by solving

$$x\left(a(t,x)\frac{\partial u}{\partial x}(t,x) + \frac{\partial u}{\partial t}(t,x) + b(t,x)u(t,x)\right) = 0.$$

The first-order linear PDE

$$a(t,x)\frac{\partial u}{\partial x}(t,x) + \frac{\partial u}{\partial t}(t,x) + b(t,x)u(t,x) = 0 \tag{3.5}$$

can be solved by the method of characteristics. The characteristic lines of (3.5) are of the form $\xi_t = (t, \varphi_t(x_0))$ with initial data $(0, x_0)$. Note that these characteristic lines depend smoothly on y and are well defined for every y. Along ξ_t, equation (3.5) becomes the linear (parametric with parameter y) ODE

$$\dot{u} = -\tilde{b}(t)\, u\,,$$

where $\tilde{b}(t) = b(\xi_t)$. Now we can define $u(\xi_t) = e^{-\int_0^t \tilde{b}(\tau)d\tau}$. This formula, being applied to all characteristics, defines a smooth solution to equation (3.5). In particular $u(t,0) = 1$ since $b(t,0) = 0$.

We have constructed a time-dependent change of coordinates $\psi(t,x)$ such that

$$\psi(t,\cdot)\circ\ \overrightarrow{\exp}\int_0^t a(\tau,\cdot)\frac{\partial}{\partial x}\ d\tau \circ \psi(t,\cdot)^{-1} = e^{t\alpha x\frac{\partial}{\partial x}}\ , \quad \text{for every } t\in[0,1]\,.$$

Recall that $\overrightarrow{\exp}\int_0^1 a(\tau,\cdot)\frac{\partial}{\partial x}\ d\tau = \varphi$. Therefore

$$\begin{aligned}\varphi &= \psi(1,\cdot)^{-1}\circ e^{\alpha x\frac{\partial}{\partial x}}\circ\psi(1,\cdot)\\ &= e^{\psi(1,\cdot)_*\alpha x\frac{\partial}{\partial x}}\ .\end{aligned}$$

Hence, we provide the desired exponential representation for every of the functions $\varphi_1,\dots,\varphi_d$ from Lemma 3.7 and the Proposition follows. □

3.4 Proof of Theorem 3.1

Let

$$\mathrm{Gr}_S\mathcal{F}_q = \{P\in \mathrm{Gr}_S\mathcal{F} : P(q) = q\}, \quad q\in M.$$

Lemma 3.8. *Any $q\in M$ possesses a neighborhood $U_q\subset M$ such that the set*

$$\left\{P\big|_{U_q} : P\in\mathrm{Gr}_S\mathcal{F}_q\right\} \tag{3.6}$$

contains a neighborhood of the identity in $C_q^\infty(U_q,M)$.

Proof. Recall that, according to the Orbit Theorem of Sussmann, the transitivity of the action of $\mathrm{Gr}\mathcal{F}$ on M implies that

$$T_qM = \mathrm{span}\{\mathrm{Ad}\,Pf(q)\ :\ P\in\mathrm{Gr}\mathcal{F},\ f\in\mathcal{F}\}.$$

Take $X_i = \mathrm{Ad}\,P_if_i$ for $i=1,\dots,d$ with $P_i\in\mathrm{Gr}\mathcal{F}$ and $f_i\in\mathcal{F}$ in such a way that $X_1(q),\dots,X_d(q)$ form a basis of T_qM. Then, for all smooth functions $a_1,\dots,a_d$, vanishing at q, the diffeomorphism

$$e^{a_1X_1}\circ\cdots\circ e^{a_dX_d} = P_1\circ e^{(a_1\circ P_1)f_1}\circ P_1^{-1}\circ\cdots\circ P_d\circ e^{(a_d\circ P_d)f_d}\circ P_d^{-1},$$

belongs to the group $\mathrm{Gr}_S\mathcal{F}_q$. By Proposition 3.6 the set (3.6) contains an open subset of $C_q^\infty(U_q,M)$, say $\mathcal{A}$. Now consider $P_0\big|_{U_q}\in\mathcal{A}$, then $P_0^{-1}\circ\mathcal{A}$ is a neighborhood of the identity contained in (3.6). □

Lemma 3.9. *Let $\mathcal{O}$ be a neighborhood of the identity in* $\mathrm{Diff}\,M$*. Then for any $q \in M$ and any neighborhood $U_q \subset M$ of q, we have*

$$q \in \mathrm{int}\,\{P(q) : P \in \mathcal{O} \cap \mathrm{Gr}_S\mathcal{F},\ \mathrm{supp}\,P \subset U_q\}\,.$$

Proof. Consider d vector fields $X_1, \ldots, X_d$ as in the proof of Lemma 3.8 and let $b \in C^\infty(M)$ be a cut-off function such that $\mathrm{supp}\,b \subset U_q$ and $q \in \mathrm{int}\,b^{-1}(1)$. Then the diffeomorphism

$$Q(s_1, \ldots, s_d) = e^{s_1 bX_1} \circ \cdots \circ e^{s_d bX_d}\,,$$

belongs to $\mathcal{O} \cap \mathrm{Gr}_S\mathcal{F}$ for any d-uple of real numbers $(s_1, \ldots, s_d)$ sufficiently close to 0. Moreover $\mathrm{supp}\,Q(s_1, \ldots, s_d) \subset U_q$. On the other hand, the map

$$(s_1, \ldots, s_d) \mapsto Q(s_1, \ldots, s_d)(q)\,,$$

is a local diffeomorphism in a neighborhood of 0. □

Next Lemma is due to Palis and Smale (see [29, Lemma 3.1]).

Lemma 3.10. *Let $\bigcup_j U_j = M$ be a covering of M by open subsets and let $\mathcal{O}$ be a neighborhood of identity in* $\mathrm{Diff}\,M$*. Then the group* $\mathrm{Diff}_0 M$ *is generated by the subset* $\{P \in \mathcal{O} : \exists\, j \text{ such that } \mathrm{supp}\,P \subset U_j\}$.

Proof. The group $\mathrm{Diff}_0(M)$ is an path–connected topological group. Therefore it is generated by any neighborhood of the identity $\mathcal{O}$.

Since M is compact we can assume that the covering $\{U_j\}$ is finite, namely $U_1 \cup \cdots \cup U_k = M$. Now let $P \in \mathcal{O}$ and consider the isotopy $H : M \times [0,1] \to M$ such that $H(0, \cdot) = I$ and $H(1, \cdot) = P$. Consider a partition of unity

$$\{\lambda_j : M \to \mathbb{R} \,|\, \mathrm{supp}\lambda_j \subset U_j\}$$

subordinated to the covering $\{U_j\}_{j=1}^k$. Let $\mathrm{supp}\lambda_j = \overline{V_j}$ and let $\mu_j : M \to M \times [0,1]$ the map $\mu_j = (I, \lambda_1 + \cdots + \lambda_j)$. Consider $Q_j = H \circ \mu_j$, then $Q_k = P$ and $Q_j = Q_{j-1}$ on $M \backslash V_j$. Finally, setting $P_j = Q_j \circ Q_{j-1}^{-1}$, we have $P = P_k \circ \cdots \circ P_1$ and $\mathrm{supp}P_j \subset U_j$. Lemma is proved. □

Proof of Theorem 3.1. According to Lemma 3.10, it is sufficient to prove that, for every $q \in M$, there exist a neighborhood $U_q \subset M$ and a neighborhood of the identity $\mathcal{O} \subset \mathrm{Diff}(M)$ such that any diffeomorphism $P \in \mathcal{O}$, whose support is contained in U_q, belongs to $\mathcal{P}$. Moreover, Lemma 3.9 allows to assume that $P(q) = q$. Finally, Lemma 3.8 completes the proof. □

CHAPTER 4

Generalized implicit function theorems

In this Chapter we briefly present the Nash–Moser technique. For more details we refer to the paper by Hamilton [19] that contains a great number of examples and counterexamples useful to understand the result. We also suggest [28, Chapter 6] for a detailed proof of Theorem 4.11. The conjugacy problem is treated in great details together with a lot of other applications in the pioneering paper by Moser [26]. We are also based on the book [7] which contains applications to small divisors problems.

The structure of the chapter is as follows. In Section 4.1 we introduce the Nash–Moser technique and we explain why we need this technique for our purposes. Section 4.2 contains the definition of the cathegory in which the Nash–Moser technique works. We also show tools useful for the computations of Chapter 5. In Section 4.3 a first statement of a generalized inverse function theorem due to Hamilton is provided. Then we briefly sketch the idea of the technique. Finally, in Section 4.4 we show how the Nash–Moser techniques works in a particular conjugacy problem. We use this example to introduce a statement of the implicit function theorem by Zehnder with weaker hypotheses. The results and the definitions in this Chapter are quoted in a slightly more general form then we strictly need for our purpose and many of them are only sketched.

4.1 Motivation

Remark 4 shows that classical implicit function theorem does not apply for the exponential from $\operatorname{Vec} S^1$ to $\operatorname{Diff}(S^1)$ essentially because these spaces are not normed spaces. Indeed, the derivative of an operator in Fréchet spaces may be invertible at one point but not at other points arbitrarily nearby, while in Banach this would have followed automatically. The exponential map on S^1 is an example of this fact. Although the derivative of the exponential map is the identity at the origin, it fails to be invertible at nearby points as the following remark shows.

Remark 6. Let us compute the differential of the exponential map at the vector field $v\partial_x$ in the direction $w\partial_x$. By definition we have

$$De^v w = \frac{\partial}{\partial \varepsilon} e^{(v+\varepsilon w)\partial_x}\big|_{\varepsilon=0} ,$$

and, using variation formula 1.9,

$$\begin{aligned} De^v w &= \frac{\partial}{\partial\varepsilon}\ \overrightarrow{\exp}\int_0^1 e^{t\mathrm{ad}vX}\varepsilon w\partial_x\,dt\big|_{\varepsilon=0}\circ e^{v\partial_x}, \\ &= \frac{\partial}{\partial\varepsilon}\ \overrightarrow{\exp}\int_0^1 e^{tvX}\varepsilon w e^{t\mathrm{ad}v\partial_x}\partial_x\,dt\big|_{\varepsilon=0}\circ e^{v\partial_x}, \\ &= \int_0^1 e^{tv\partial_x} w\,\mathrm{Ad}\,e^{tv\partial_x}\partial_x\,dt\circ e^{v\partial_x}. \end{aligned}$$

Now consider the case in which $v = 2\pi/k$, with k integer. Then the exponential of the constant vector field $v\partial_x$ is a rotation through an angle $2\pi/k$. Moreover, since v is constant, then $\mathrm{Ad}\,e^{tv\partial_x}\partial_x = \partial_x$. Therefore

$$De^v w(x) = \int_0^1 w\left(x + \frac{2\pi t}{k}\right) dt\ \partial_x \circ e^{\frac{2\pi k}{t}\partial_x}.$$

Note that this derivatives cannot be surjective since it annihilates the functions $w(x) = \sin kx$ and $w(x) = \cos kx$. More generally, each term in the Fourier series expansion of w is multiplied by some constant, and for these terms that constant is zero. Thus the derivative of the exponential map at a small rotation through an angle $2\pi/k$ is never invertible.

Moreover it is not possible to look at the exponential as a map between the Banach spaces C^k. This is due to the “loss of derivatives”. Indeed, while the exponential maps C^k vector fields into C^k diffeomorphisms, its differential has an unbounded right inverse, as we will show in Section 5.2.3, since the inverse maps the space C^k into C^{k-1}.

4.2 The Nash–Moser cathegory

We define a category of “tame” Fréchet spaces and “tame” nonlinear maps, which is essentially due to Sergeraert.

Through this Section the letter C denotes a strictly positive constant, whose value may change from line to line.

Definition 13. A *graded Fréchet space* is a Fréchet space with a family of seminorms $\|\cdot\|_n$ increasing in strength, namely

$$\|f\|_0 \le \|f\|_1 \le \|f\|_2 \le \dots .$$

Definition 14. We say that two gradings $\|\cdot\|_n$ and $\|\cdot\|'_n$ are *tamely equivalent* of degree r and base n_0 if

$$\|f\|_n \le C\|f\|'_{n+r}\,, \qquad \text{and} \qquad \|f\|'_n \le C\|f\|_{n+r}$$

for all $n > n_0$ (with a constant C which may depend on n).

Remark 7. All of the definitions and theorems in this chapter are still valid when a grading is replaced by a tamely equivalent one.

An example of tamely equivalent gradings is defined in Section 5.2.1.

Definition 15. We say that a linear map $L : X \to Y$ of one graded space into another satisfies a *tame estimate* of degree r and base n_0 if

$$\|Lf\|_n \leq C\|f\|_{n+r}$$

for each $n \geq n_0$ (with a constant C which may depend on n). We say that L is *tame* if it satisfies a tame estimate for some r and n_0.

Note that a tame linear map is automatically continuous in the Fréchet space topologies.

Given a Banach space $\mathcal{B}$ we define $\Sigma(\mathcal{B})$ as the space of exponential decreasing sequences, namely the space of all sequences $\{f_k\}$ of elements of $\mathcal{B}$ such that

$$\|\{f_k\}\|_n = \sum_{k=0}^{\infty} e^{nk}\|f_k\| < \infty\,,$$

for all $n \geq 0$. Then $\Sigma(\mathcal{B})$ is a graded space. This definition is needed to introduce the tame spaces.

Definition 16. We say that a graded space X is a *tame space* if there exist a Banach space $\mathcal{B}$ and tame linear maps $L_1 : X \to \Sigma(\mathcal{B})$ and $L_2 : \Sigma(\mathcal{B}) \to X$ such that the composition L_1L_2 is the identity.
In this case we say that X is a *tame direct summand* of $\Sigma(\mathcal{B})$.

This definition is necessary to guarantee the existence of a *smoothing operator* on the space X. Indeed, smoothing operators are particularly easy to construct on the model space $\Sigma(\mathcal{B})$. A smoothing operator on a tame space X is a family of linear mappings $S_t : X \to X$ such that for all $m \leq n$ we have

$$\|S_tf\|_n \leq Ce^{(n-m)t}\|f\|_m\,, \tag{4.1}$$

$$\|(\mathrm{Id} - S_t)f\|_m \leq Ce^{(m-n)t}\|f\|_n\,, \tag{4.2}$$

where C may depend on n and m. On the space $\Sigma(\mathcal{B})$ it can be constructed by taking a smooth function $s(t)$ such that: $s(t) = 0$ for $t \leq 0$, $s(t) = 1$ for $t \geq 1$, and $0 \leq s(t) \leq 1$ in between. Then, for every sequence $f = \{f_k\} \in \Sigma(\mathcal{B})$, setting

$$(S_tf)_k = s(t-k)f_k\,,$$

we have the required family of operators. We will see in Section 4.3 that the proof of Nash–Moser involves smoothing operators to overcome the difficulties due to the loss of derivatives.

We can now define tame estimates for a nonlinear map.

Definition 17. Let X and Y be tame spaces and $T : U \subset X \to Y$. We say that T satisfies *tame estimates* of degree r and base n_0 if

$$\|T(f)\|_n \leq C(\|f\|_{n+r} + 1),$$

for all $f \in U$ and all $n \geq n_0$. We say that T is a *(smooth) tame map* if T is smooth and all its derivatives satisfy tame estimates. Note that the constant C may depend on n and we allow r, n_0, and C to vary from neighborhood to neighborhood.

A useful property of tame maps is the following

Proposition 4.1. *A composition of tame maps is tame.*

From now on we focus on the setting of our problem.

Proposition 4.2. *If M is a compact manifold then $C^\infty(M)$ and* $\operatorname{Vec} M$ *are tame spaces.*

The group of diffeomorphisms carries a similar structure being a Lie group. A tame Fréchet manifold is a manifold modeled on a tame Fréchet space whose coordinate transition functions are smooth tame maps. In analogy with the definition of Lie group we have that a *tame Lie group* is a smooth tame Fréchet manifold with a group structure such that the composition and the inverse map are smooth tame maps.

Proposition 4.3. *If M is a compact manifold then* $\operatorname{Diff}(M)$ *is a tame Fréchet manifold. The map $(P, Q) \in \operatorname{Diff}(M) \times \operatorname{Diff}(M) \mapsto P \circ Q^{-1} \in \operatorname{Diff}(M)$ is a smooth tame map. Hence* $\operatorname{Diff}(M)$ *is a tame Lie group.*

We conclude with a list of properties of smooth maps on a manifold.

Proposition 4.4. (Interpolation Inequalities) *Let M compact manifold and $\ell \leq m \leq n$, then, for all $f \in C^\infty(M)$, we have*

$$\|f\|_m^{n-\ell} \leq C\|f\|_n^{m-\ell}\|f\|_\ell^{n-m}. \tag{4.3}$$

It is important to remark that the interpolations inequalities are a consequence of the existence of a smoothing operator. In fact, these inequalities holds true also replacing $C^\infty(M)$ by any tame space.
For a pair of functions we have the following corollary.

Corollary 4.5. *Let $f, g \in C^\infty(M)$, if (i, j) lies on the segment joining (k, ℓ) and (m, n) then there exists a constant C independent on f and g, such that*

$$\|f\|_i\|g\|_j \leq C(\|f\|_k\|g\|_\ell + \|f\|_m\|g\|_n).$$

Since for smooth functions holds

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k} f D^k g, \tag{4.4}$$

then, we have, as a corollary, that exists C such that:

$$\|fg\|_n \leq C(\|f\|_n \|g\|_0 + \|f\|_0 \|g\|_n).$$

Another important tool used in next chapter is the Iterated chain rule formula also known as "Faá di Bruno's Formula"(see for example [31]). Let f, g be smooth functions from U and $V \subset \mathbb{R}^d$ respectively to $\mathbb{R}^d$ such that the $U \subset g(V)$. Then

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{m_1!\, m_2! \, \cdots \, m_n!} f^{(m_1+\cdots+m_n)}(g(x)) \prod_{j=1}^{n} \left(\frac{g^{(j)}(x)}{j!} \right)^{m_j}, \tag{4.5}$$

where the sum is over all the n-uples $(m_1, \ldots, m_n)$ such that $1m_1 + 2m_2 + \cdots + nm_n$. As a consequence of this formula and of the Interpolation inequalities we have the following proposition.

Proposition 4.6. (Composition is tame) *If there exists K such that $\|f\|_1 \leq K$ and $\|g\|_1 \leq K$ then*

$$\|f \circ g\|_n \leq C(\|f\|_n + \|g\|_n + 1). \tag{4.6}$$

By "Faá di Bruno's Formula" directly follows that

$$\sup_x |D^n f^k(x)| \leq C \|f\|_n \|f\|_0^{k-1}.$$

And if $f \in C^\infty(U)$ is such that there exist $0 < \alpha < \beta$ such that $\alpha < f(x) < \beta$ for all $x \in U$ then

$$\sup_x \left| D^n \frac{1}{f(x)} \right| \leq C \|f\|_n \tag{4.7}$$

where C depends on n, α, β, U.

Finally, we state some results on tame maps

Proposition 4.7. *Let $U \subset \mathbb{R}^d$, then the map $f \in C^\infty(U) \mapsto e^f \in C^\infty(U)$ is tame of degree* 0.

This proposition can be easily generalized for the exponential map from $\operatorname{Vec} U$ to $\operatorname{Diff}_0(U)$.

Corollary 4.8. *Let $U \subset \mathbb{R}^d$, then the exponential map* $\exp : \operatorname{Vec} U \to \operatorname{Diff}(U)$ *is tame.*

An analogous result is valid for the chronological exponential.

Proposition 4.9. *Let $U \subset \mathbb{R}^d$ be bounded and consider*

$$\mathcal{U} = \{V_t \in \operatorname{Vec} U, t \in [0,1] \,:\, \|V_t\|_1 < 1\}\,.$$

Then for every $n > 0$ there exist $C > 0$ such that

$$\| \overrightarrow{\exp} \int_0^t V_\tau \, d\tau \|_n \leq C(\|V_t\|_n + 1)\,,$$

for every $V_t \in \mathcal{U}$.

Proof. Let, for simplicity, $P^t := \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$. By definition

$$q \circ P^t = q + \int_0^t q \circ P^\tau \circ V_\tau \, d\tau\,, \quad q \in B\,. \tag{4.8}$$

Let us prove the statement by induction over k. For $n = 0$ the statement follows from (4.8). Let $n > 0$ and consider the iterated chain rule (4.5)

$$\begin{aligned} \|P^t\|_n &= 1 + \int_0^t \|P^\tau \circ V_\tau\|_n d\tau \\ &\leq 1 + t(\|V_t\|_1 \|P^t\|_n + C \sum_{m_1,\dots,m_{n-1}} \|V_t\|_{r_{n-1}} \|P^t\|_1^{m_1} \cdots \|P^t\|_n^{m_{n-1}})\,, \end{aligned}$$

where $r_{n-1} = m_1 + \cdots + m_{n-1}$. Then

$$(1 - t\|V_t\|_1)\|P^t\|_n \leq 1 + tC \sum_{m_1,\dots,m_{n-1}} \|V_t\|_{r_{n-1}} \|P^t\|_1^{m_1} \cdots \|P^t\|_n^{m_{n-1}}\,.$$

Now, using interpolation inequalities and since $\|P^t\|_0 \leq C(\|V_t\|_0 + 1)$, for every term of the sum we have

$$\|V_t\|_{m_1+\cdots+m_{n-1}} \|P^t\|_1^{m_1} \cdots \|P^t\|_n^{m_{n-1}} \leq C(\|V_t\|_n + \|P^t\|_{n-1}).$$

Since $1 - t\|V_t\|_1 = \alpha > 0$, then

$$\begin{aligned} \|P^t\|_n &\leq (1 + tC(\|V_t\|_n + \|P^t\|_{n-1}))/\alpha \\ &\leq C(\|V_t\|_n + \|P^t\|_{n-1})\,. \end{aligned}$$

Then the statement follows from the inductive hypotesis. □

4.3 An inverse function theorem by Nash and Moser

We can now state Hamilton's version of the Nash–Moser inverse function theorem.

Theorem 4.10. *Let X and Y be tame spaces and $F : U \subset X \to Y$ a smooth tame map. Suppose that the equation for the derivative $DF(u)h = k$ has a solution $h = DF(u)^{-1}k$ for all u in U and all k, and that the family of inverses $DF^{-1} : U \times Y \to X$ is a smooth tame map. Then F is locally surjective. Moreover in a neighborhood of any point F has a smooth tame right inverse.*

The proof of the Theorem is very technical and it is beyond the scope of this thesis to expose it in details. Anyway we sketch the idea that is very elegant and lies in the construction of a rapidly convergent sequence of approximations by solving the linearized equation. The first step consists in proving that it is not restrictive to assume that $0 \in U$ and $F(0) = 0$. Then the idea is to replace the tame spaces X and Y by the model spaces $\Sigma(\mathcal{B}_1)$ and $\Sigma(\mathcal{B}_2)$ for some Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$. Roughly speaking, it is like working in the scale of Banach spaces

$$C^0 \supset C^1 \supset \ldots \supset C^n \supset \ldots$$

instead of C^∞, in this sense we exploit the definition of tame space. Denote by $L(u) = DF(u)^{-1}$, then the iteration scheme is as follows.

$$\begin{cases} u_{n+1} &= u_n - S_{t_n} L(u_n) F(u_n)\,, \\ u_0 &= 0\,. \end{cases} \tag{4.9}$$

Since at every step the inverse L loses derivatives then the iterates will belong to a larger space in the scale of Banach spaces $\Sigma(\mathcal{B}_1)$. Tame estimates guarantees that there is an upper bound on the number of lost derivatives, then the smoothing S_{t_n} allows to catch up with the loss of derivatives to the detriment of the accuracy of the approximation. The task is then to find an appropriate sequence of numbers t_n depending on the degree of F and L, in order to make the scheme to converge.

Although Hamilton's statement is very elegant it is not the best possible. Hypotheses are too strong and, in fact, the Nash–Moser method allows to assume weaker hypotheses as we will show in the next section.

4.4 A conjugacy problem and the implicit function theorem with quadratic error

Theorem 4.10 impose a very strong condition on the differential of map F. Indeed, the essential requirements is that

$$F'(u)h = k \tag{4.10}$$

admits solution for every u in a open set U. This condition is rather stringent and often it is possible to ensure the solvability of the equation in a particular point $u_0 \in U$ but not in the whole set U. In fact, in some case the Nash–Moser method works with the weaker condition of the existence of an "approximate right inverse".

For a tame map $F : X \times Y \to Z$, where X, Y, and Z are tame spaces the definition of approximate right inverse is as follows.

Definition 18. We say that D_xF has an *approximate right inverse* if , for every $(x,y) \in U \subset X \times Y$ there exist a smooth tame map $L(x,y)h$ linear in h,

$$L : U \times Z \to X$$

and a smooth tame map $Q(x,y)\{h,k\}$ bilinear in h and k,

$$Q : U \times Z \times Z \to X$$

such that for all $(x,y) \in U$ and all $z \in Z$ we have

$$D_xF(x,y)L(x,y)z = z + Q(x,y)\{F(x,y),z\}\,.$$

Note that L is required to be a precise right inverse only at points (x,y) that satisfy $F(x,y) = 0$.

We conjecture that our problem of stability of the exponential map, sketched in Section 5.3, is in some sense similar to the one treated below. We show a conjugacy problem, due to Moser [26], in which (4.10) is not solvable but the iterative scheme works. We also show the construction of an approximate right inverse. The problem, which is firstly successfully treated by Siegel [38], can be stated as follows: given analytic functions f and Φ sufficiently nearby, find a conformal mapping u such that

$$u^{-1} \circ f \circ u = \Phi\,.$$

More in general, consider a differentiable group action

$$\begin{array}{rccc} F: & X \times G & \to & X \\ & (f,u) & \mapsto & F(f,u)\,, \end{array}$$

of a group G, with null element I, on an infinite dimensional manifold X, such that F satisfies the "conjugacy identities", i.e.

$$\left\{ \begin{array}{lcl} F(f,I) & = & f \\ F(f,u \circ u_0) & = & F(F(f,u),u_0)\,. \end{array} \right. \tag{4.11}$$

Note that (4.11) are clearly satisfied by $F(f,u) = u^{-1} \circ f \circ u$. The problem is to solve the equation

$$F(f,u) = \Phi\,,$$

for given f and Φ.

Set $u_0 = I$ and assume that $u_1, \ldots, u_n$ have already been constructed. Then we set

$$u_{n+1} = u_n \circ v\,.$$

Call $\lambda_n(v) = u_{n+1}$, so that $\lambda_n(I) = u_n$ and $\lambda_0(v) = v$. We want to find $v = I + \hat{v}$ such that

$$F(f, u_n \circ v) = \Phi\,, \tag{4.12}$$

up to terms linear in $\hat{v}$ and in the error $F(f, u_n \circ v) - \Phi$. Now let

$$f_n = F(f, u_n)\,,$$

then, by hypothesis f_n is close to Φ, and by (4.11) we have

$$\begin{aligned} F(f, u_{n+1}) &= F(f, u_n \circ v) \\ &= F(F(f, u_n), v) \\ &= F(f_n, v)\,. \end{aligned} \tag{4.13}$$

Then (4.12) takes the form

$$F(f_n, v) = \Phi\,.$$

Formally expanding the left-hand side at the pair (Φ, I) we have

$$F(\Phi, I) + F_f(\Phi, I)(f_n - \Phi) + F_u(\Phi, I)\hat{v} = \Phi\,. \tag{4.14}$$

Now, by (4.11) and since

$$F_f(\Phi, I)g = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(\Phi + \varepsilon g, I) - F(\Phi, I)) = g,$$

then (4.14) reduces to

$$F_u(\Phi, I)\hat{v} = \Phi - f_n\,. \tag{4.15}$$

Therefore we have an equation for $\hat{v}$, that, if solved, defines the next step u_{n+1}. We have a rapidly convergent sequence of approximations. At least formally, the iteration scheme converges quadratically. Indeed, if, in some norm, the error $f_n - \Phi$ is of order ε_n then, by (4.15), $\hat{v}$ is of order ε_n too. Moreover we determined $\hat{v}$ from equation (4.12) which is satisfied up to terms linear in $\hat{v}$ and in the error $f_n - \Phi$. Thus, the error at the step $n+1$ will be of order ε_n^2.

Note that the construction of the scheme requires invertibility of $F_u(\Phi, I)$ only and not of $F_u(f, u)$. In fact, this is the case of the existence of an approximate right inverse, which can be determined starting from $F_u(\Phi, I)^{-1}$. Indeed, consider the relation (4.13) and consider, for the left-hand side, the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(f, \lambda_n(I + \varepsilon\hat{v})) - F(f, \lambda_n(I))) = F_u(f, u_n)\lambda_n'(I)\hat{v}\,. \tag{4.16}$$

On the other hand we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(f_n, (I + \varepsilon\hat{v})) - F(f_n, I)) = F_u(f_n, I)\,,$$

and by means of the Taylor formula, since by hypotesis $f_n - \Phi$ is small, we get

$$F_u(f_n, I) = F_u(\Phi, I)\hat{v} + Q(f_n - \Phi, \hat{v})\,, \tag{4.17}$$

for some bilinear operator Q. Therefore by (4.16) and (4.17) we get

$$F_u(f, u_n)\lambda_n'(I)\hat{v} = F_u(\Phi, I)\hat{v} + Q(f_n - \Phi, \hat{v})\,,$$

in other words we have that, at the point u_n of the iteration scheme, $F_u(f, u_n)$ has an approximate right inverse, that is

$$\lambda_n'(I)(F_u(\Phi, I))^{-1}\,.$$

This is only a guiding principle in dealing with this kind of problem. We have been vague about topology, convergences, and regularity. The ideas given can be formalized in the following theorem due to Zehnder (see [28, Chapter 6]), which, in fact, generalizes an inverse function theorem of the Nash–Moser type by Schwartz (see [35, Chapter II]).

Theorem 4.11. *Let X, Y, and Z be tame Fréchet spaces and let F be a smooth tame map defined on an open set U in $X \times Y$ to Z,*

$$F : U \subset X \times Y \to Z\,.$$

Suppose that, for every $(x, y) \in U$, $D_x F(x, y)$ has an approximate right inverse $L(x, y)$. Then if

$$F(x_0, y_0) = 0\,,$$

for some $(x_0, y_0) \in U$ we can find neighborhoods of x_0 and y_0 such that for all y in the neighborhood of y_0 there exists an x in the neighborhood of x_0 with $F(x, y) = 0$. Moreover the solution $x = f(y)$ is defined by a smooth tame map f.

CHAPTER 5

An application of the Nash–Moser method to the exponential map

In this chapter we give an alternative proof of Proposition 3.6. The proof is an application of the Nash–Moser inverse function theorem and the technical part of the proof, i.e. the tame estimates, are, for the sake of readability, given is Section 5.2. We conclude in the last section, Section 5.3, stating future developments of the problem arising from the additional informations about the product of exponential maps of Proposition 3.6 that such an alternative proof gives us. The results in this chapter are part of our paper [9].

5.1 Invertibility of the differential of the exponential map

In this section we are going to prove the following restatement of Proposition 3.6.

Proposition 5.1. *Let $X_i \in \mathrm{Vec}\mathbb{R}^d,\ i = 1, \dots, d$, such that*

$$\mathrm{span}\{X_1(0), \dots, X_d(0)\} = \mathbb{R}^d.$$

Then, there exist $\varrho > 0$ and an open subset $\mathcal{U} \subset C_0^\infty(B_\varrho)^d$, such that the mapping

$$\begin{array}{rcl} F : \mathcal{U} & \to & C_0^\infty(B_\varrho)^d, \\ (a_1, \dots, a_d) & \mapsto & \left(e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d}\right)\big|_{B_\varrho}, \end{array} \tag{5.1}$$

is an open map from $\mathcal{U}$ into $C_0^\infty(B_\varrho)^d$, where

$$B_\varrho = \left\{ e^{s_1 X_1} \circ \cdots \circ e^{s_d X_d}(0) : |s_i| < \varrho,\ i = 1, \dots, d \right\}.$$

In order to apply Theorem 4.10 we need to check, for the map F the following points:

1. $DF(a)[\xi]$ is a tame map both in $a \in \mathcal{U}$ and $\xi \in C_0^\infty(B_\varrho)$;

2. $DF(a)$ has a right inverse for every $a \in \mathcal{U}$;

3. the right inverse of DF is a tame map.

Indeed, by Corollary 4.8 and Proposition 4.1, map F is tame, so it remains to check estimates only for the differential and its inverse. These estimates must be proved directly and, since this is a very technical part of the proof, for convenience to the reader, these are given in Section 5.2. Proof strategy splits into four main steps. Since we have to find an inverse of the differential of F in the whole set $\mathcal{U}$, the first step consists in finding a "good" set $\mathcal{U}$. In the second step we prove that it is not restrictive to consider the problem along a single direction X_i, for every $i = 1, \dots, d$, turning it into a one dimensional problem with parameters. In Lemma 5.2 we find a tame change of coordinates that linearizes the vector field $a_i X_i$ and, finally, we prove the invertibility of the differential of F.

First of all, since $\mathrm{span}\{X_1(0), \dots, X_d(0)\} = \mathbb{R}^d$ then there exists $\varrho > 0$ such that

$$X_i(q) \neq 0, \quad \text{for all } q \in B_\varrho, \quad i = 1, \dots, d.$$

Now, let $w_1, \dots, w_d \in C_0^\infty(B_\varrho)$ such that

$$\langle d_q w_i, X_i(q) \rangle = -1, \quad \text{for all } q \in B_\varrho, \ i = 1, \dots, d,$$

then take

$$\mathcal{U} = \bigoplus_{i=1}^{d} \{a \in C_0^\infty(B_\varrho) \, : \, \|a - \varepsilon w_i\|_1 < \delta, \|a\|_2 < \gamma\}, \tag{5.2}$$

where $\delta < \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2\|X_1\|_0}, \dots, \frac{\varepsilon}{2\|X_d\|_0}\}$. Let us call $U_1, \dots, U_d$ the sets that compose the direct sum (5.2).
Note that, for every $\gamma > 0$, if $\varepsilon < \min\{\frac{\gamma}{\|w_1\|_2}, \dots, \frac{\gamma}{\|w_d\|_2}\}$, then $\mathcal{U}$ is an open nonempty subset of $C_0^\infty(B_\varrho)^d$.

Let us start with the computation of the differential of F. Call $\phi_i(a_i) = e^{a_i X_i}$ for $i = 1, \dots, d$. So

$$F(a) = \phi_1(a_1) \circ \dots \circ \phi_d(a_d)\,.$$

Now let us compute the differential of ϕ_i, for every $i = 1, \dots, d$. Since this computation is the same for every i we omit the subscript. We have

$$\left(\frac{\partial}{\partial a} e^{aX}\right) : \xi \mapsto \left(\int_0^1 e^{-\int_0^t \langle da, X \rangle \circ e^{\tau a X}\, d\tau} \xi \circ e^{taX}\, dt\right) X \circ e^{aX}.$$

Indeed

$$\begin{aligned} D\phi(a)[\xi] &= \frac{\partial}{\partial \varepsilon} e^{(a+\varepsilon\xi)X}\big|_{\varepsilon=0}, \\ &= \frac{\partial}{\partial \varepsilon} \overrightarrow{\exp} \int_0^1 e^{t\mathrm{ad}aX} \varepsilon\xi X \, dt\big|_{\varepsilon=0} \circ e^{aX}, \\ &= \frac{\partial}{\partial \varepsilon} \overrightarrow{\exp} \int_0^1 e^{taX} \varepsilon\xi e^{t\mathrm{ad}aX} X \, dt\big|_{\varepsilon=0} \circ e^{aX}, \\ &= \int_0^1 e^{taX} \xi \mathrm{Ad} e^{taX} X \, dt \circ e^{aX}, \end{aligned}$$

now the time varying vector field $\mathrm{Ad}e^{taX}X$ is the vector field X twisted by the flow of the rescaling by a smooth function a of X itself. We expect that $\mathrm{Ad}e^{taX}X$ is a time dependent rescaling of X, indeed

$$\begin{aligned}\frac{d}{dt}\mathrm{Ad}\,e^{taX}X &= \frac{d}{dt}e^{t\,\mathrm{ad}\,aX}X\\ &= e^{t\,\mathrm{ad}\,aX}(\mathrm{ad}\,aX)X\\ &= e^{t\,\mathrm{ad}\,aX}[aX,X]\\ &= -e^{taX}\langle da,X\rangle e^{t\,\mathrm{ad}\,aX}X\\ &= -e^{taX}\langle da,X\rangle\,\mathrm{Ad}\,e^{taX}X\,,\end{aligned}$$

then

$$\mathrm{Ad}e^{taX}X = \left(e^{-taX}\right)_* X = e^{-\int_0^t \langle da,X\rangle\circ e^{\tau aX}\,d\tau}X.$$

Let us call

$$A(a)\xi = \int_0^1 e^{-\int_0^t \langle da,X\rangle\circ e^{\tau aX}\,d\tau}\xi\circ e^{taX}\,dt, \tag{5.3}$$

so

$$D\phi(a)[\xi] = A(a)\xi X\circ\phi(a).$$

Let $a=(a_1,\ldots,a_d)$, $\xi=(\xi_1,\ldots,x_d)$, then

$$\begin{aligned}DF(a)[\xi] &= A(a_1)\xi_1X_1\circ\phi_1(a_1)\circ\cdots\circ\phi_d(a_d)+\\ &\quad +\phi_1(a_1)\circ A(a_2)\xi_2X_2\circ\phi_2(a_2)\circ\cdots\circ\phi_d(a_d)+\\ &\quad +\cdots+\\ &\quad +\phi_1(a_1)\circ\cdots\circ A(a_d)\xi_dX_d\circ\phi_d(a_d).\end{aligned} \tag{5.4}$$

Equation (5.4) can be written also as:

$$\begin{aligned}DF(a)[\xi] &= [A(a_1)\xi_1X_1+\mathrm{Ad}(\phi_1(a_1))A(a_2)\xi_2X_2+\cdots\\ &\quad +\mathrm{Ad}(\phi_1(a_1)\circ\cdots\circ\phi_{n-1}(a_{n-1}))A(a_d)\xi_dX_d]\circ F(a)\\ &= \sum_{i=1}^d u_iY_i\circ F(a)\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}u_1 &= A(a_1)\xi_1\\ u_2 &= \phi_1(a_1)A(a_2)\xi_2\\ &\vdots\\ u_d &= \phi_1(a_1)\circ\cdots\circ\phi_{d-1}(a_{d-1})A(a_d)\xi_d\end{aligned}$$

and

$$\begin{aligned} Y_1 &= X_1 \\ Y_2 &= \phi_1(a_1)\circ X_2 \circ \phi_1(a_1)^{-1} \\ &\vdots \\ Y_d &= \phi_1(a_1)\circ\cdots\circ\phi_{d-1}(a_{d-1})\circ X_d\circ\phi_{d-1}(a_{d-1})^{-1}\circ\cdots\circ\phi_1(a_1)^{-1}. \end{aligned}$$

Now we want to solve

$$D\phi(a)[\xi] = V\,, \tag{5.6}$$

for every given vector field $V \in \mathrm{Vec}\, B_\varrho$ and $a \in \mathcal{U}$. For γ sufficiently small the vector fields $Y_1, \dots, Y_d$ are linearly independent at 0 and V can be written as $V = \sum_{i=1}^d v_i Y_i$. Therefore solving (5.6) is equivalent to solve for every $\eta_i \in C^\infty(B_\varrho)$ the equation

$$A(a_i)\xi_i = \eta_i, \quad a_i \in U_i, \tag{5.7}$$

for every $i = 1, \dots, d$.

Next step consists in finding a coordinates system on B_ϱ such that the vector field $a_i X_i$ is linear. Since the argument does not depend on $i = 1, \dots, d$, from now on the subscript i is omitted. For every $a \in U$ we have $a(0) = 0$. Moreover $\langle d_q a, X(q)\rangle < 0$ for every $q \in B_\varrho$, indeed

$$\begin{aligned} \langle d_q a, X(q)\rangle &= \langle d_q(a-\varepsilon w), X(q)\rangle + \varepsilon\langle d_q w, X(q)\rangle \\ &\le \|a-\varepsilon w\|_1 \|X\|_0 - \varepsilon \\ &< -\varepsilon/2. \end{aligned}$$

Therefore X is transversal to $a^{-1}(0)$ at every point. In particular we may rectify the field X in such a way that, in new coordinates,

$$X = \frac{\partial}{\partial x_1}, \quad \text{and} \quad a(0, x_2, \dots, x_d) = 0\,. \tag{5.8}$$

Moreover, since this system of coordinates depends only on an orthogonality relation with the differential of a, then it depends tamely, with degree at most 1, on a.
In order to simplify notation we set $x = x_1$ and $y = (x_2, \dots, x_d)$.
The following lemma allows us to consider only the linear part of the field $a\frac{\partial}{\partial x}$. Below we prove only existence of the change of variables. The proof that it is tame is done in Section 5.2.1.

Lemma 5.2. *Let $a \in U$, then there exists a smooth change of coordinates Ψ on B_ϱ that linearizes the vector field $a(x,y)\frac{\partial}{\partial x}$.*
Moreover Ψ is a tame map with respect to a with tame inverse.

Proof. Since $a(0,y) = 0$, then we can write $a(x,y) = -x\alpha(y) + xb(x,y)$, with $b(0,y) = 0$. Consider a solution $x(t)$ of the parametric ODE $\dot{x} = a(x,y)$. We look for a diffeomorphism

$$\Psi(x,y) = (\psi(x,y), y) \tag{5.9}$$

such that if $z = \psi(x,y)$ then

$$\dot{z} = -\alpha(y)z.$$

Suppose that $\psi(x,y) = x + x\phi(x,y)$, with $\phi(0,y) = 0$, then

$$\begin{aligned}
\frac{d}{dt}z &= \frac{d}{dt}(x + x\phi(x,y)) \\
&= \dot{x} + \dot{x}\phi(x,y) + x\dot{x}\frac{\partial\phi}{\partial x}(x,y) \\
&= a(x,y)(1 + \phi + x\frac{\partial\phi}{\partial x}(x,y)),
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\frac{d}{dt}z &= -\alpha(y)z \\
&= -\alpha(y)x(1 + \phi(x,y)).
\end{aligned}$$

Therefore ϕ is the solution of the following family of ODE with parameter y

$$\begin{aligned}
\frac{\partial\phi}{\partial x}(x,y) &= -\frac{b(x,y)}{a(x,y)}(1 + \phi(x,y)), \\
\phi(0,y) &= 0.
\end{aligned}$$

So

$$\phi(x,y) = e^{-\int_0^x \frac{b(s,y)}{a(s,y)}ds} - 1,$$

and

$$\psi(x,y) = xe^{-\int_0^x \frac{b(s,y)}{a(s,y)}ds}. \tag{5.10}$$

We have an explicit formula for the change of coordinates Ψ. $\square$

Last Lemma allows us to assume that $aX = -\alpha(y)x\frac{\partial}{\partial x}$, where $-\alpha(y) = \dfrac{\partial a}{\partial x}(0,y)$. Hence

$$\xi \circ e^{taX}(x,y) = \xi(e^{-t\alpha(y)}x, y)\,,$$

which implies

$$\int_0^1 e^{\int_0^t e^{-\tau\alpha(y)x\frac{\partial}{\partial x}}\alpha(y)d\tau}\xi(e^{-t\alpha(y)}x,y)dt = \int_0^1 e^{t\alpha(y)}\xi(e^{-t\alpha(y)}x,y)dt. \tag{5.11}$$

Call

$$\hat{A}(\xi) = \int_0^1 e^{t\alpha(y)} \xi(e^{-t\alpha(y)}x, y) dt. \tag{5.12}$$

In this last step we want to prove that this map has a smooth family of right inverses. Moreover, in Section 5.2 we prove that $\hat{A}$ is a tame map (see Subsection 5.2.2) with a tame family of right inverses (Subsection 5.2.3).
Let

$$\xi(x, y) = \xi(0, y) + x\xi_x(0, y) + x^2 u(x, y),$$

then

$$\hat{A}(\xi(0, y)) = \frac{e^{\alpha}(y) - 1}{\alpha(y)} \xi(0, y),$$

and

$$\hat{A}(x\xi_x(0, y)) = x\xi_x(0, y).$$

Now let

$$v(x, y) = \frac{1}{x} \int_0^x u(s, y) ds,$$

and

$$R : v(x, y) \mapsto e^{-\alpha(y)} v(e^{-\alpha(y)}x, y),$$

then

$$\begin{aligned}
\hat{A}(x^2 u(x, y)) &= x^2 \int_0^1 e^{-\alpha(y)t} u(e^{-t\alpha(y)}x, y)\, dt \\
&= \frac{x^2}{\alpha(y)} \int_{e^{-\alpha(y)}}^1 u(\tau x, y)\, d\tau \\
&= \frac{x^2}{\alpha(y)} (v(x, y) - e^{-\alpha(y)} v(e^{-\alpha(y)}x, y)) \\
&= \frac{x^2}{\alpha(y)} (I - R) v(x, y).
\end{aligned}$$

Let $\|v\|_{C^{n,0}} = \sup_{1 \le i \le n} \left\| \frac{\partial^i v}{\partial x^i} \right\|_{C^0}$. Since $\alpha(y)$ is uniformly bounded away from 0, it is clear that R is a contraction from the space $C^{\infty,0}$ of continuous functions smooth with respect to x into itself. Hence $(I - R)$ is invertible in this space and $(I - R)^{-1} = \sum_{k=0}^{\infty} R^k$ maps a function f, smooth on the box, in a continuous function $g = (I - R)^{-1} f$ smooth with respect to x. We want to prove that if $f \in C^\infty$ then $g = (I - R)^{-1} f \in C^\infty$. Let us do it by induction. Suppose that $g \in C^{\infty,0}$ then

$$D_y g = (I - R)^{-1} (f_y + \alpha' e^{-\alpha} g + \alpha' e^{-2\alpha} g_x) \in C^{\infty,0},$$

so $g \in C^{\infty,1}$.
Now let $n \geq 1$ and suppose that $g \in C^{\infty,n-1}$, then

$$\begin{aligned}
D_y^n f(x,y) &= D_y^n (I-R) g \\
&= D_y^n g(x,y) - D_y^n (e^{-\alpha(y)} g(e^{-\alpha(y)}x,y)) \\
&= D_y^n g(x,y) + \sum_{k=0}^{n} \binom{n}{k} D_y^k e^{-\alpha(y)} D_y^{n-k} g(e^{-\alpha(y)}x,y) \\
&= (I-R) D_y^n g(x,y) + \sum_{k=1}^{n} \binom{n}{k} D_y^k e^{-\alpha(y)} D_y^{n-k} g(e^{-\alpha(y)}x,y),
\end{aligned}$$

so we have

$$D_y^n g(x,y) = (I-R)^{-1} \left(D_y^n f(x,y) - \sum_{k=1}^{n} \binom{n}{k} D_y^k e^{-\alpha(y)} D^{n-k} g(e^{-\alpha(y)}x,y) \right),$$

which is $C^{\infty,0}$ by hypotesis. Therefore $g \in C^{\infty,n}$.
Therefore we have that $\hat{A}$ is invertible and we have an explicit formula for the inverse of $\hat{A}$. By

$$\hat{A}(\xi) = \eta,$$

we get

$$\begin{aligned}
u(x,y) &= \frac{\partial}{\partial x} \left(x(I-R)^{-1} \frac{\alpha(y)}{x^2} (\eta(x,y) - \eta(0,y)) \right) \\
&= \frac{\alpha(y)}{x^2} \sum_{k=0}^{\infty} G_k(e^{-\alpha(y)k}x,y)\,,
\end{aligned} \tag{5.13}$$

where

$$G_k(x,y) := e^{\alpha(y)k}(x\eta_x(x,y) - \eta(x,y) + \eta(0,y))\,,$$

then

$$\hat{A}^{-1}(\eta) = \frac{\alpha(y)}{e^{\alpha}(y) - 1} \eta(0,y) + x\eta_x(0,y) + \alpha(y) \sum_{k=0}^{\infty} G_k(e^{-\alpha(y)k}x,y). \tag{5.14}$$

This completes the proof of Proposition 5.1.

5.2 Tame Estimates

First, let us fix some notation used through this section. We consider coordinates $(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Given, $B \subset \mathbb{R}^d$ and $f \in C^\infty(B)$ we denote by $D^n f$ the n-th differential of f. Moreover we set

$$D_x^n f := f^{(n,0)} := \frac{\partial^n}{\partial x^n} f\,,$$

and

$$D_y^n f := f^{(0,n)} := \frac{\partial^n}{\partial y^n} f \,.$$

We will choose at every line the more efficient and brief notation from the two above. Note that, in order to simplify notation, we denote the differential of a function with respect to y treating y as a 1-dimensional variable. Finally, recall that the letter C denotes a strictly positive constant, whose value may change from line to line.

5.2.1 Tame estimates for the change of coordinates

Here we prove that the change of coordinates Ψ defined in (5.9) and (5.10) is a tame map. Clearly, it is sufficient to find tame estimates for

$$\psi = xe^{-\int_0^x \frac{b(s,y)}{a(s,y)}ds} \,.$$

Since exponential is tame of degree 0, then

$$\|\psi\|_n' = \left\|\frac{\psi}{x}\right\|_n \leq C \left\| \int_0^x \frac{b(s,y)}{a(s,y)} ds \right\|_n ,$$

where

$$\|f\|_n' = \|f/x\|_n$$

is a tamely equivalent granding on $C_0^\infty(B_\varrho)$.
Now, for the product of the smooth function, the first term of sum (4.4) is

$$\begin{aligned} \left| D_y^n \int_0^x \frac{b(s,y)}{a(s,y)} ds \right| &= \left| D_y^n \int_0^x \frac{b_x(\nu(s),y)}{-\alpha(y)+b(s,y)} \right| , \\ &\leq C \left(\|b_x\|_n \left\| \frac{1}{-\alpha+b} \right\|_0 + \|b_x\|_0 \left\| \frac{1}{-\alpha+b} \right\|_n \right) . \end{aligned}$$

Since

$$-\alpha(y) + b(x,y) + \varepsilon \leq \|a - \varepsilon w\|_1 < \varepsilon/2$$

then $-\alpha(y) + b(x,y) < -\varepsilon/2$ and so $|-\alpha(y) + b(x,y)|$ is uniformly bounded away from 0. Therefore, applying property (4.7),

$$\begin{aligned} \sup \left| D_y^n \int_0^x \frac{b(s,y)}{a(s,y)} ds \right| &\leq C\left(\|b\|_{n+1} + \|-\alpha + b\|_n\right) \\ &\leq C(\|a\|_{n+1}' + 1) \,. \end{aligned}$$

The remaining terms are of the form $D_y^{n-k}D_x^k$ with $k \geq 1$. So

$$\begin{aligned}\left|D_y^{n-k}D_x^k\int_0^x \frac{b(s,y)}{a(s,y)}ds\right| &= \left|D_y^{n-k}D_x^{k-1}\frac{b(x,y)}{a(x,y)}\right| \\ &\leq \left\|\frac{b(x,y)}{a(x,y)}\right\|_{n-1} \\ &\leq C\left(\left\|\frac{b(x,y)}{x}\right\|_{n-1} + \left\|\frac{1}{-\alpha+b}\right\|_{n-1}\right). \end{aligned} \tag{5.15}$$

Now, by Taylor expansion of b in x, we obtain

$$D_x^q\frac{b(x,y)}{x} = (-1)^q\frac{q!}{x^q}\sum_{\ell=0}^{q}(-1)^\ell\frac{b^{(q+1,0)}(\nu_\ell(x),y)x^{q+1}}{(q-\ell+1)!\ell!},$$

so that

$$\sup\left|D_y^pD_x^q\frac{b(x,y)}{x}\right| \leq C\sup|b^{(q+1,p)}(x,y)|\,.$$

Hence

$$\left\|\frac{b(x,y)}{x}\right\|_{n-1} \leq C\|b\|_n\,.$$

Therefore by (5.15) and (4.7) we get the tame estimates

$$\left|D_y^{n-k}D_x^k\int_0^x\frac{b(s,y)}{a(s,y)}ds\right| \leq C(\|a\|_n'+1)\,,$$

and we prove that Ψ is a tame map of degree 1.

Now let us prove that the inverse is also tame. Let $g(x,y)$ be a smooth function such that

$$\psi(g(x,y),y) = x. \tag{5.16}$$

It is sufficient to prove tame estimates for g. First of all note that

$$\begin{aligned}|\psi_x(x,y)| &= \frac{\alpha(y)}{\alpha(y)-b(x,y)}e^{-\int_0^x\frac{b(s,y)}{a(s,y)}ds} \\ &\geq \frac{\varepsilon-\delta}{\varepsilon+\delta}e^{-\varrho\frac{\delta}{\varepsilon-\delta}}\end{aligned}$$

for every $(x,y)\in B_\varrho$. Hence

$$\|1/\psi_x\|_0 \leq C_1, \tag{5.17}$$

where C_1 depends only on ε and ϱ. Differentiating (5.16) with respect to x we have

$$g_x(x,y) = \frac{1}{\psi_x(g(x,y),y)}\,.$$

Then we have an upper bound for g, indeed

$$\begin{aligned} |g(x,y)| &\leq |x| \sup |g_x(x,y)| \\ &\leq \varrho C_1 =: C_2. \end{aligned}$$

Moreover $\|\psi\|_1 \leq C\|a\|_2 \leq C_3$. Now let $n > 1$, by iterated chain rule (4.5) we have

$$\begin{aligned} & D^n\psi(g(x,y),y) = \\ \psi_x(g(x,y),y)D^n(g(x,y),y) &+ \sum_{\Pi} c_k\, D^k\psi\big|_{(g(x,y),y)} \prod_{j=1}^{k-1} \left(D^j(g(x,y),y)\right)^{m_j}, \end{aligned}$$

where the sum is over the set Π of all the $(n-1)$-uples $(m_1,\dots,m_{n-1})$ such that $m_1+\cdots+(n-1)m_{n-1} = n$ and where we denote for simplicity $k = m_1+\cdots+m_{n-1}$. On the other hand, differentiating (5.16) n times, we have

$$D^n\psi(g(x,y),y) = 0.$$

Therefore

$$\begin{aligned} \|(g(x,y),y)\|_n &\leq C\|1/\psi_x\|_0 \,\|\sum D^k\psi\big|_{(g(x,y),y)} \prod_{j=1}^{k-1} \left(D^j(g(x,y),y)\right)^{m_j} \|_0 \\ &\leq C\sum_{\Pi} \|\psi\|_k \|g\|_1^{m_1} \dots \|g\|_{n-1}^{m_{n-1}}. \end{aligned}$$

By interpolation inequalities (4.3) we have, for every $j = 1,\dots,n-1$,

$$\|g\|_j^{m_j} \leq C\|g\|_0^{\frac{n-j-1}{n-1}m_j} \|g\|_{n-1}^{\frac{j-1}{n-1}m_j}$$

and also

$$\|\psi\|_k \leq C\|\psi\|_1^{\frac{n-k}{n-1}} \|\psi\|_n^{\frac{k-1}{n-1}}.$$

Hence,

$$\begin{aligned} \|\psi\|_k\|g\|_1^{m_1}\dots\|g\|_{n-1}^{m_{n-1}} &\leq C\left(\|g\|_0^{\frac{nk-n-1}{n-1}} \|g\|_{n-1}^{\frac{n-k}{n-1}} \|\psi\|_1^{\frac{n-k}{n-1}} \|\psi\|_n^{\frac{k-1}{n-1}}\right) \\ &\leq C\|g\|_0^{\frac{n(k-1)-k}{n-1}} \left(\|g\|_0\|\psi\|_n\right)^{\frac{k-1}{n-1}} \left(\|g\|_{n-1}\|\psi\|_1\right)^{\frac{n-k}{n-1}} \\ &\leq C\|g\|_0^{\frac{n(k-1)-k}{n-1}} \left(\|g\|_0\|\psi\|_n + \|g\|_{n-1}\|\psi\|_1\right). \end{aligned}$$

Then, by the bounds on $\|g\|_0$ and $\|\psi\|_1$, there exists a constant C, depending only on ϱ, γ, and ε such that

$$\|g\|_n \leq C(\|\psi\|_n + \|g\|_{n-1}).$$

By induction and using the fact that ψ is tame we have that g is a tame map and so is the inverse of ψ.

5.2.2 Tame estimates for $\hat{A}$

Here we give tame estimates for

$$\hat{A}(\xi) = \int_0^1 e^{t\alpha(y)}\xi(e^{-t\alpha(y)}x, y)dt\,.$$

We have

$$\begin{aligned} |D^n\hat{A}(\xi(x,y))| &= |D^n\int_0^1 e^{t\alpha(y)}\xi(e^{-t\alpha(y)}x, y)dt| \\ &\le \int_0^1 |D^n e^{t\alpha(y)}\xi(e^{-t\alpha(y)}x, y)|dt \\ &\le C\int_0^1 \sum_{j=0}^{n} |D^j\xi(e^{-t\alpha(y)}x, y)||D^{n-j}e^{t\alpha(y)}|. \end{aligned}$$

Set $h(x,y) = (e^{-\alpha(y)t}, y)$, since the exponential is tame (Proposition 4.7), we have $\|h\|_j \le C\|\alpha\|_j$, for every j. In particular, $\|h\|_1$ is bounded by a constant independent of α. Therefore, by (4.6), on every open subset of $C^\infty(B_\varrho)$ of the form $\|\xi\|_0 \le K$, we obtain

$$\begin{aligned} \|\hat{A}(\xi)\|_n &\le C\sum_{j=0}^{n}(\|\xi\|_j + \|h\|_j\|\xi\|_1)\|\alpha\|_{n-j} \\ &\le C\sum_{j=0}^{n}(\|\xi\|_j + \|\alpha\|_j\|\xi\|_1)\|\alpha\|_{n-j} \\ &\le C(\|\xi\|_n + \|\alpha\|_n\|\xi\|_1)\,, \end{aligned}$$

and $\hat{A}$ is a tame linear map of degree 0.

In particular $\hat{A}$ does not lose derivatives, that is $\hat{A}$ maps a C^k function ξ in a C^k function $\hat{A}(\xi)$. It is natural to ask whether there is a "gain of derivatives". Although $\hat{A}$ is an integral operator there is no gain of derivatives, since there is no gain with respect to y.

Note that, to determine the number of derivatives lost or gained by the differential map $DF(a)$, for every fixed $a \in \mathcal{U}$, it is sufficient to study the map $\hat{A} : \xi \in C^\infty \mapsto \hat{A}(\xi) \in C^\infty$. Indeed the rectification in (5.8) and the change of coordinates of Lemma 5.2 do not depend on ξ and do not affect the loss or gain of derivatives of $DF(a)$.

5.2.3 Tame estimates for $\hat{A}^{-1}$

In order to prove that $\hat{A}^{-1}$ is tame let us verify tame estimates for the first term of (5.14). Note that

$$\left\|\frac{\alpha(y)}{e^{\alpha(y)} - 1}\right\|_0 \le C.$$

Moreover by inequality (4.7),

$$\left\| \frac{1}{e^{\alpha(y)} - 1} \right\|_n \leq \| e^{\alpha(y)} - 1 \|_n \leq \|\alpha\|_n,$$

therefore

$$\left\| \frac{\alpha(y)}{e^{\alpha(y)} - 1} \right\|_n \leq \|\alpha\|_n,$$

finally, the first term is tame since

$$\left\| \frac{\alpha(y)}{e^{\alpha}(y) - 1} \eta(0, y) \right\|_n \leq C(\|\alpha\|_n + \|\eta\|_n).$$

For the second term, we have

$$\begin{aligned} \|x\eta_x(0,y)\|_n &= \sup_{1 \leq j \leq n} \|x\eta^{(1,j)}(0,y) + \eta^{(1,j-1)}(0,y)\|_0 \\ &\leq C\|\eta\|_{n+1} \,. \end{aligned}$$

Now consider the last addend. Note that, using Lagrange formula twice we have

$$\begin{aligned} \|G_k(e^{-\alpha(y)k}x, y)\|_0 &= \|x(\eta_x(e^{-\alpha(y)k}x, y) - \eta_x(\nu_k(x,y), y))\|_0 \\ &\leq \|x^2 e^{-\alpha(y)k}\|_0 \ \|\eta_{xx}\|_0 \\ &\leq \varrho^2 e^{-\frac{\varepsilon}{2}k} \|\eta\|_2 \,, \end{aligned}$$

therefore

$$\| \sum_{k=0}^{\infty} G_k(e^{-\alpha(y)k}x, y)\|_0 \leq C\|\eta\|_2.$$

Now,

$$\|\alpha(y) \sum_{k=0}^{\infty} G_k(e^{-\alpha(y)k}x, y)\|_n \leq C \left(\|\alpha\|_n \ \|\eta\|_2 + \left\| \sum_{k=0}^{\infty} G_k(e^{-\alpha(y)k}x, y) \right\|_n \|\alpha\|_0 \right),$$

then it remains to estimate only the quantity

$$\left\| \sum_{k=0}^{\infty} G_k(e^{-\alpha(y)k}x, y) \right\|_n .$$

For every n, k we have

$$|D^n e^{\alpha(y)k}| \leq C\|\alpha\|_n k^n e^{\alpha(y)k}, \tag{5.18}$$

and

$$|D^n e^{-\alpha(y)k}| \leq C\|\alpha\|_n k^n e^{-\alpha(y)k}, \tag{5.19}$$

which implies

$$|D^n(e^{-\alpha(y)k}, y)| \le C\|\alpha\|_n k^n e^{-\alpha(y)k}. \tag{5.20}$$

Moreover, the following estimates hold

$$\begin{aligned}
|D_y^r G_k(x,y)| &\le \sum_{p=0}^{r} \binom{r}{p} \left|D_y^{r-p} e^{\alpha(y)k}\right| \left|D_y^p(x\eta_x(x,y) - \eta(x,y) + \eta(0,y))\right| \\
&\le C \sum_{p=0}^{r} \|\alpha\|_{r-p} k^{r-p} e^{\alpha(y)k} |x\eta^{(1,p)}(x,y) - \eta^{(0,p)}(x,y) + \eta^{(0,p)}(0,y)| \\
&\le C \sum_{p=0}^{r} \|\alpha\|_{r-p} k^{r-p} e^{\alpha(y)k} |x(\eta^{(1,p)}(x,y) - \eta^{(1,p)}(\nu_k(x,y),y))| \\
&\le Ck^r e^{\alpha(y)k} |x| (\|\alpha\|_{r+1}\|\eta\|_0 + \|\alpha\|_0\|\eta\|_{r+1}) , \qquad (5.21)
\end{aligned}$$

and therefore

$$|D_y^r G_k(x,y)|\big|_{(e^{-\alpha(y)k}x,y)} \le Ck^r(\|\alpha\|_{r+1} + \|\eta\|_{r+1}).$$

Now, let $j \ge 1$ and consider:

$$\begin{aligned}
|D_y^{r-j} D_x^j G_k(x,y)| &= \left|D_y^{r-j}\, e^{\alpha(y)k}\, D_x^{j-1}(x\eta_{xx}(x,y))\right| \\
&= |D_y^{r-j} e^{\alpha(y)k}(x\eta^{(j+1,0)}(x,y) + (j-1)\eta^{(j,0)}(x,y))| \\
&\le C \sum_{p+q=r-j} |D_y^q e^{\alpha(y)k}|\, |x\eta^{(j+1,p)}(x,y) + (j-1)\eta^{(j,p)}(x,y)| \\
&\le Ck^{r-j}(1 + (j-1)e^{\alpha(y)k})(\|\alpha\|_{r+1}\|\eta\|_0 + \|\alpha\|_0\|\eta\|_{r+1}) . \qquad (5.22)
\end{aligned}$$

By (4.5), if $m_1 + \cdots + nm_n = n$ and $r = m_1 + m_2 + \cdots + m_n$, then

$$|D^n G_k(e^{-\alpha(y)k}x, y)| \le \sum_{\Pi} |D_y^r G_k(x,y)\Big|_{(e^{-\alpha(y)k}x,y)}| \prod_{j=1}^{n} \left|D^j(e^{-\alpha(y)k}, y)\right|^{m_j}$$

and, by (5.21) and (5.22),

$$|D^n G_k(e^{-\alpha(y)k}x, y)| \le \sum_{\Pi} Ck^r e^{-\alpha(y)kr}(\|\alpha\|_{r+1}\|\eta\|_0 + \|\eta\|_{r+1}\|\alpha\|_0) \prod_{j=1}^{n} \|\alpha\|_j^{m_j}.$$

Interpolation inequalities (4.3) imply

$$\|\alpha\|_{r+1} \le C\|\alpha\|_1^{\frac{n-r}{n}} \|\alpha\|_{n+1}^{\frac{r}{n}} , \tag{5.23}$$

and, for $j = 1, \ldots, n$,

$$\|\alpha\|_j \le C\|\alpha\|_1^{\frac{n+1-j}{n}} \|\alpha\|_{n+1}^{\frac{j-1}{n}} , \tag{5.24}$$

since $\|\alpha\|_1$ is bounded and, if $\|\eta\|_0 \le C$, then

$$\|\alpha\|_{r+1}\|\eta\|_0\|\alpha\|_1^{m_1}\cdots\|\alpha\|_n^{m_n} \le C\|\alpha\|_{n+1}.$$

In a similar way we can prove

$$\|\eta\|_{r+1} \le C\|\eta\|_1^{\frac{n-r}{n}}\|\eta\|_{n+1}^{\frac{r}{n}},$$

which implies, together with (5.24), that

$$\begin{aligned}\|\eta\|_{r+1}\|\alpha\|_0\|\alpha\|_1^{m_1}\cdots\|\alpha\|_n^{m_n} &\le C(\|\eta\|_{n+1})^{\frac{r}{n}}(\|\eta\|_1\|\alpha\|_{n+1})^{\frac{n-r}{n}}\\ &\le C(\|\eta\|_{n+1}+\|\eta\|_1\|\alpha\|_{n+1}),\end{aligned}$$

Therefore, finally,

$$\begin{aligned}\|\sum_{k=0}^{\infty}G_k(e^{-\alpha(y)k}x,y)\|_n &\le \sum_{k=0}^{\infty}\|G_k(e^{-\alpha(y)k}x,y)\|_n\\ &\le \sum_{k=0}^{\infty}\left\|\sum D_y^r G_k(x,y)\big|_{(e^{-\alpha k}x,y)}\prod_{j=1}^{n}\left(D^j(e^{-\alpha(y)k},y)\right)^{m_j}\right\|_0\\ &\le C\sum_{k=0}^{\infty}k^n\|e^{-\alpha(y)k}\|_0(\|\eta\|_{n+1}+\|\alpha\|_{n+1}\|\eta\|_1)\\ &\le C\sum_{k=0}^{\infty}k^n e^{-\frac{\epsilon}{2}k}(\|\eta\|_{n+1}+\|\alpha\|_{n+1}\|\eta\|_1)\\ &= C(\|\eta\|_{n+1}+\|\alpha\|_{n+1}\|\eta\|_1).\end{aligned}$$

This completes the proof that $\hat{A}^{-1}$ is a tame map.

Note that $\hat{A}^{-1}$ lose exactly 1 derivative in ξ. Indeed there is loss of 1 derivative with respect to x that is due to the differential operator $\frac{\partial}{\partial x}$ in (5.13).

5.3 Open problems about small perturbations of the exponential map

In this section we sketch the ideas of what will be part of our future investigations in [10]. We want to study if the exponential map F in (5.1) remains locally onto under some kind of small perturbations. Again, the problem arises in control theory. Indeed we know by Proposition 5.1 that there exist $B \subset \mathbb{R}^d$ and $\mathcal{O} \subset \mathrm{Diff}_0(B)$ such that every $P \in \mathcal{O}$ can be written as

$$P = e^{a_1X_1}\circ\cdots\circ e^{a_dX_d},$$

for some $a_1, \ldots, a_d \in \mathcal{U} \subset C^\infty(B)$, where $X_1, \ldots, X_d$ are vector fields linearly independent at $0 \in B$. By Orbit Theorem we know that, if $\mathcal{F} = \{f_1, \ldots, f_m\}$ is a bracket generating family of vector field on B, then $X_i = \mathrm{Ad}\, P^i f_{j_i}$, with $P^i \in \mathrm{Gr}\mathcal{F}$. Therefore

$$\begin{aligned} P &= e^{a_1 X_1} \circ \cdots \circ e^{a_d X_d} \\ &= P^1 \circ e^{(P^1)^{-1}(a_1) f_{j_1}} \circ (P^1)^{-1} \circ \cdots \circ P^d \circ e^{(P^d)^{-1}(a_d) f_{j_d}} \circ (P^d)^{-1} \\ &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^d u_i(t) b_i f_{j_i} + \sum_{i=1}^m v_i(t) f_i \, dt \,, \end{aligned}$$

where $b_i = (P^i)^{-1}(a_i)$. Then, Proposition 5.1 implies local surjectivity of the map

$$F : b \in \mathcal{U}' \mapsto \overrightarrow{\exp} \int_0^1 \sum_{i=1}^d u_i(t) b_i f_{j_i} + \sum_{i=1}^m v_i(t) f_i \, dt \,, \tag{5.25}$$

where

$$\mathcal{U}' = \bigoplus_{i=1}^d (P^i)^{-1}(U_i)$$

for $U_1, \ldots, U_d$ defined by (5.2). In terms of control–affine systems this means that for every $P \in \mathcal{O}$ there exists time–varying feedback control $w_1(t,q), \ldots, w_m(t,q)$ piecewise constant with respect to $t \in [0,1]$, such that P is the flow at time 1 of system

$$\dot{q} = \sum_{i=1}^m w_i(t,q) f_i(q) \,.$$

Moreover we know that the dependence on q of the controls is, in fact, a linear dependence in the functions $b_i(q)$.

Remark 8. Actually, by Corollary 3.3, we know that the result holds true for any given diffeomorphisms in the connected component of the identity not just for the one in the open subset $\mathcal{O}$. But Proposition 5.1 gives us the additional information that the exponential map (5.25) has also an invertible differential if $b \in \mathcal{U}'$.

It is natural, as we did in Section 2.4, to ask whether it is possible to assume the time–varying feedback control $w_1(t,q), \ldots, w_m(t,q)$ to be trigonometric polynomials. Let

$$\begin{aligned} F(b) &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^d u_i(t) b_i f_{j_i} + \sum_{i=1}^m v_i(t) f_i \, dt \\ &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{d+m} u_i(t) b_i f_{j_i} \, dt \,, \end{aligned}$$

provided that $b_{d+1} = \ldots = b_{d+m} = 1$. Consider the truncated fourier series of $u_i(t)$, say $u_i^n(t)$. Then

$$u_i^n \to u_i, \quad \text{as } n \to \infty\,,$$

in $L^1[0,1]$. Consider

$$F_n(b) = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{d+m} u_i^n(t) b_i f_{j_i}\, dt\,,$$

then for every $b \in \mathcal{U}'$ by Lemma 1.6,

$$F_n(b) \to F(b), \quad \text{as } n \to \infty$$

in the C^∞ topology. The problem is to determine if there exists n such that F_n is locally onto. The idea is to find an approximate right inverse in order to apply Zehnder version of Nash–Moser implicit function theorem, Theorem 4.11. First of all note that $F_n(b)$ is a tame map with the same degree of $F(b)$. Now let $r_i^n(t) = u_i(t) - u_i^n(t)$ and call

$$V_t^n = \sum_{i=1}^{d+m} r_i^n(t) b_i f_{j_i}\,,$$

we have

$$V_t^n(b) \to 0, \quad \text{as } n \to \infty\,,$$

in $L^1[0,1]$ and uniformly with all derivatives in $q \in B$. By variation formula (1.9),

$$\begin{aligned} F_n(b) &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{d+k} u_i(t) b_i f_{j_i} - V_t^n\, dt \\ &= \overrightarrow{\exp} \int_0^1 \operatorname{Ad} F^t(b) V_t^n(b)\, dt \circ F(b) \\ &= R_n(b) \circ F(b)\,, \end{aligned}$$

where

$$R_n(b) \to \mathrm{Id}, \quad \text{as } n \to \infty\,.$$

in the C^∞ topology.

We now compute the differential of F_n at a point $b \in \mathcal{U}'$ applied to the d-uple of smooth functions $\xi = (\xi_1, \ldots, \xi_d)$. Using (1.10),

$$\begin{aligned} D_b F_n \xi &= \int_0^1 \operatorname{Ad} F_n^t(b) \sum_{i=1}^{d} u_i(t) \xi_i f_{j_i}\, dt \circ F^n(b) \\ &= \int_0^1 \operatorname{Ad}(R_n^t(b) \circ F^t(b)) \sum_{i=1}^{d} u_i(t) \xi_i f_{j_i}\, dt \circ R_n(b) \circ F(b) \\ &= \int_0^1 \operatorname{Ad} R_n^t(b) \circ \operatorname{Ad} F^t(b) \sum_{i=1}^{d} u_i(t) \xi_i f_{j_i}\, dt \circ F(b) \circ \operatorname{Ad} F(b)^{-1} R_n(b)\,, \end{aligned}$$

now we have, by definition,

$$R_n(b) = \mathrm{Id} + \int_0^1 R_n^t(b) \circ \mathrm{Ad}\, F^t(b) V_t^n(b)\, dt\,, \tag{5.26}$$

and

$$\mathrm{Ad}\, R_n^t(b) = \mathrm{Id} + \int_0^t \mathrm{Ad}\, R_n^\tau(b) \circ \mathrm{ad}(\mathrm{Ad}\, F^\tau(b) V_\tau^n(b))\, d\tau\,, \tag{5.27}$$

therefore the differential of F_n can be written as

$$\begin{aligned} D_b F_n \xi \;=\; & D_b F \xi \\ & + \int_0^1 \Big(\int_0^t \mathrm{Ad}\, R_n^\tau(b) \circ \mathrm{ad}(\mathrm{Ad}\, F^\tau(b) V_\tau^n(b))\, d\tau \Big) \circ \mathrm{Ad}\, F^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i}\, dt \circ F^n(b) \\ & + \int_0^1 \mathrm{Ad}\, F^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i}\, dt \circ \Big(\int_0^1 R_n^t(b) \circ \mathrm{Ad}\, F^t(b) V_t^n(b)\, dt \Big) \circ F(b)\,, \end{aligned}$$

or, equivalently,

$$\begin{aligned} D_b F_n \xi \;=\; & D_b F \xi \circ \mathrm{Ad}\, F(b)^{-1} R_n(b) \\ & + \int_0^1 \Big(\int_0^t \mathrm{Ad}\, R_n^\tau(b) \circ \mathrm{ad}(\mathrm{Ad}\, F^\tau(b) V_\tau^n(b))\, d\tau \Big) \circ \mathrm{Ad}\, F^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i}\, dt \circ F^n(b)\,. \end{aligned}$$

In other words, the differential of F_n is, up to small perturbation, an invertible linear operator. It remains to study how the perturbation acts on the linear operator in order to determine whether $D_b F_n$ has an approximate right inverse or not. It is remarkable the particular dependence of $D_b F_n$ on b, indeed b appears only in $F^t(b)$ and $V_t^n(b)$. We believe that this property of the differential allows us to follow an argument similar to the one showed in Section 4.4 about a conjugacy problem by Moser.

Finally, this problem leads to a great number of other related open problems, such as, to mention just two of the closest, extend the result in order to reach every diffeomorphisms in the connected component and study whether the result holds true also for a control–affine system with drift. We believe that the last problem can be done in a similar way, that is, studying how small perturbation, such as a small drift, affects the (right) invertibility of the exponential map.

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