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Families of Vector Fields Which Generate the Group of Diffeomorphisms

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Abstract—Given a compact manifold M and a family of vector fields $\mathcal{F}$ such that the group generated by $\mathcal{F}$ acts transitively on M, we prove that the group of all diffeomorphisms of M that are isotopic to the identity is generated by the exponentials of vector fields in $\mathcal{F}$ rescaled by smooth functions.

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INTRODUCTION

In this paper we give a simple sufficient condition for a family of flows on a smooth compact manifold M to generate the group $\mathrm{Diff}_0(M)$ of all diffeomorphisms of M that are isotopic to the identity.

If all flows are available, then the result follows from the simplicity of the group $\mathrm{Diff}_0(M)$ (see [9]). Indeed, flows are just one-parametric subgroups of $\mathrm{Diff}_0(M)$ and all one-parametric subgroups generate a normal subgroup. In other words, any diffeomorphism of M isotopic to the identity can be presented as a composition of the exponentials of smooth vector fields.

The problem of realizing a diffeomorphism as a composition of the exponentials (i.e. flows at time 1) of smooth vector fields arises in the framework of control theory. In this framework, in the interesting cases, the system cannot evolve along all the possible directions but only along a prescribed vector distribution. The result proved in this paper holds for a proper subset of the space of smooth vector fields on M. Our main result is as follows.

Theorem. *Let* $\mathcal{F} \subset \mathrm{Vec}\, M$ *be a family of smooth vector fields and let* $\mathrm{Gr}\,\mathcal{F} = \{e^{t_1 f_1} \circ \ldots \circ e^{t_k f_k} \colon t_i \in \mathbb{R},\ f_i \in \mathcal{F},\ k \in \mathbb{N}\}$.

If $\mathrm{Gr}\,\mathcal{F}$ *acts transitively on* M, *then there exists a neighborhood* $\mathcal{O}$ *of the identity in* $\mathrm{Diff}_0(M)$ *and a positive integer* m *such that every* $P \in \mathcal{O}$ *can be presented in the form*

$$P = e^{a_1 f_1} \circ \ldots \circ e^{a_m f_m}$$

for some $f_1, \ldots, f_m \in \mathcal{F}$ *and* $a_1, \ldots, a_m \in C^\infty(M)$.

In particular, if $\mathcal{F}$ is a bracket-generating family of vector fields, then any diffeomorphism in $\mathrm{Diff}_0(M)$ can be presented as a composition of the exponentials of vector fields in $\mathcal{F}$ rescaled by smooth functions. Indeed, on a connected manifold, if a distribution is completely nonholonomic or "bracket-generating," then any two points of the manifold can be connected by a curve whose velocity belongs to the distribution; in other words, the corresponding control system is completely controllable. This is the statement of the classical Rashevsky–Chow theorem.

We prove that a distribution providing controllability on M, and close under multiplication by smooth functions, also provides exact controllability on the group of diffeomorphisms of M. In fact,

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a stronger result is valid. The theorem states that every diffeomorphism sufficiently close to the identity can be presented as the composition of m exponentials, where the number m depends only on $\mathcal{F}$.

The structure of the paper is the following. In Section 1 we introduce the notation used throughout this text. Then we state some simple corollaries to the theorem. In Section 2 we prove an auxiliary result concerning local diffeomorphisms in $\mathbb{R}^d$. Given smooth vector fields $X_1, \ldots, X_d \in \operatorname{Vec} \mathbb{R}^d$ on $\mathbb{R}^d$ that are linearly independent at the origin, we find a closed neighborhood B of the origin in $\mathbb{R}^d$ such that the image of the map

$$F\colon\ (a_1, \ldots, a_d) \mapsto e^{a_1 X_1} \circ \ldots \circ e^{a_d X_d}\big|_B$$

from $C_0^\infty(B)^d$ to $\operatorname{Diff}_0(B)$ has nonempty interior. This result is achieved using the generalized implicit function theorem by Nash and Moser. In Section 3 we show how to reduce the proof of the theorem to the mentioned auxiliary fact using a geometric idea which goes back to the orbit theorem of Sussmann [7]. In Section 4 the technical part of the proof is given in great details.

1. PRELIMINARIES AND COROLLARIES

Let M be a smooth d-dimensional compact connected manifold. Throughout the paper, 'smooth' means C^∞. We denote by $\operatorname{Vec} M$ the Lie algebra of smooth vector fields on M and by $\operatorname{Diff}_0(M)$ the connected component of the identity of the group of diffeomorphisms of M. If V is a neighborhood of the origin in $\mathbb{R}^d$, we set $C_0^\infty(V) = \{a \in C^\infty(V)\colon a(0) = 0\}$. Similarly, if U is an open subset of M, then $C_q^\infty(U, M)$ is the Fréchet manifold of smooth maps $F\colon U \to M$ such that $F(q) = q$. All the spaces above are endowed with the standard C^∞ topology. The topology of the space of smooth functions $C^\infty(B)$ on an open set B is given by the family of seminorms

$$\|a\|_n \stackrel{\text{def}}{=} \sup_{1 \le |k| \le n} \sup_{x \in B} |D^k a(x)|.$$

Given an autonomous vector field $V \in \operatorname{Vec} M$, we denote by $t \mapsto e^{tV}$, with $t \in \mathbb{R}$, the flow on M generated by V, which is a one-parametric subgroup of $\operatorname{Diff}_0(M)$.

If V_τ is a nonautonomous vector field, using "chronological" notation (see [1]), we denote by $\overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ the "nonautonomous flow," at time t, of the time-varying vector field V_τ.

We denote by $\operatorname{Ad} P V$ the action of a diffeomorphism P on a vector field V, namely, $\operatorname{Ad} P V = P \circ V \circ P^{-1}$. Given $V \in \operatorname{Vec} M$, we denote $\operatorname{ad} V = \frac{d}{dt} \operatorname{Ad} P^t\big|_{t=0}$, where $P^t = e^{tV}$. Below we will use the following properties of the action: $\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau = \operatorname{Ad} \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ and, for every smooth function a, $\operatorname{Ad} P aV = P(a) \operatorname{Ad} P V$.

Given a family of vector fields $\mathcal{F} \subset \operatorname{Vec} M$, we denote by $\operatorname{Gr} \mathcal{F}$ the subgroup of $\operatorname{Diff}_0(M)$ generated by e^{tf}, $f \in \mathcal{F}$, $t \in \mathbb{R}$, and by $\operatorname{Lie} \mathcal{F}$ the Lie subalgebra of $\operatorname{Vec} M$ generated by $\mathcal{F}$. We also set $\operatorname{Lie}_q \mathcal{F} = \{V(q)\colon V \in \operatorname{Lie} \mathcal{F}\}$.

A family $\mathcal{F} \in \operatorname{Vec} M$ is called *bracket-generating*, or completely nonholonomic, if

$$\operatorname{Lie}_q \mathcal{F} = T_q M \qquad \text{for every } \ q \in M.$$

The Rashevsky–Chow theorem [8, 2] states that $\operatorname{Gr} \mathcal{F}$ acts transitively on M for any bracket-generating and symmetric family $\mathcal{F}$.

A direct consequence of the theorem is the following

Corollary 1.1. *Let $\mathcal{F} \subset \operatorname{Vec} M$. If $\operatorname{Gr} \mathcal{F}$ acts transitively on M, then*

$$\operatorname{Gr}\big\{af\colon\ a \in C^\infty(M),\ f \in \mathcal{F}\big\} = \operatorname{Diff}_0(M).$$

Another corollary, stated from a geometric viewpoint in terms of completely nonholonomic vector distributions, is the following.

Corollary 1.2. *Let $\Delta \subset TM$ be a completely nonholonomic vector distribution. Then any diffeomorphism of M isotopic to the identity has a form $e^{f_1} \circ \ldots \circ e^{f_m}$, where $f_1, \ldots, f_m$ are sections of Δ.*

We conclude by recalling a classical result, due to Lobry [4], which claims that $\mathrm{Gr}\{f_1, f_2\}$ acts transitively on M for a generic pair of smooth vector fields (f_1, f_2). Namely, the set of pairs of vector fields (f_1, f_2) such that $\mathrm{Gr}\{f_1, f_2\}$ acts transitively on M is an open dense (in the C^∞ topology) subset of the product space $\mathrm{Vec}\, M \times \mathrm{Vec}\, M$.

2. AN AUXILIARY RESULT

We start the proof of the theorem with an auxiliary lemma that is actually the main part of the proof.

Lemma 2.1 (main lemma). *Let $X_i \in \mathrm{Vec}\, \mathbb{R}^d$, $i = 1, \ldots, d$, be such that*

$$\mathrm{span}\{X_1(0), \ldots, X_d(0)\} = \mathbb{R}^d.$$

Then, there exist $\varrho > 0$ and an open subset $\mathcal{U} \subset C_0^\infty(B_\varrho)^d$ such that the mapping

$$F\colon \mathcal{U} \to C_0^\infty(B_\varrho)^d, \qquad (a_1, \ldots, a_d) \mapsto \left(e^{a_1 X_1} \circ \ldots \circ e^{a_d X_d}\right)\big|_{B_\varrho}, \tag{1}$$

is an open map from $\mathcal{U}$ into $C_0^\infty(B_\varrho)^d$, where

$$B_\varrho = \left\{e^{s_1 X_1} \circ \ldots \circ e^{s_d X_d}(0)\colon\ |s_i| < \varrho,\ i = 1, \ldots, d\right\}.$$

In order to prove this result, we are going to use Hamilton's version of the Nash–Moser inverse function theorem (see [3]), which is stated in terms of *tame maps* and *tame spaces*. The space of smooth functions on an open set is an example of a tame space (see [3, Theorem 1.3.6]) and the group of diffeomorphisms is a tame Lie group (see [3, Theorem 2.3.5]). Roughly speaking, a (smooth) tame map is a smooth map between tame spaces that cannot lose more than a certain number, called degree, of derivatives. Tame estimates (i.e. estimates on the number of lost derivatives) for the map F must be proved directly and, since this is a very technical part of the proof, for convenience of the reader, we give these estimates, together with precise definitions and preliminary results, in Section 4.

In order to apply the Nash–Moser theorem, we need to check, for the map F defined in (1), the following points:

(i) $DF(a)[\xi]$ is a tame map both in $a \in \mathcal{U}$ and $\xi \in C_0^\infty(B_\varrho)$;

(ii) $DF(a)$ has a right inverse for every $a \in \mathcal{U}$;

(iii) the right inverse of DF is a tame map.

The proof strategy splits into four main steps. Since we have to find an inverse of the differential of F in the whole set $\mathcal{U}$, the first step consists in finding a "good" set $\mathcal{U}$. In the second step we prove that it is not restrictive to consider the problem along a single direction X_i, for every $i = 1, \ldots, d$, turning it into a one-dimensional problem with parameters. In Lemma 2.2 we find a tame change of coordinates that linearizes the vector field $a_i X_i$ and, finally, we prove the invertibility of the differential of F.

Proof of Lemma 2.1. First of all, since $\mathrm{span}\{X_1(0), \ldots, X_d(0)\} = \mathbb{R}^d$, there exists $\varrho > 0$ such that

$$X_i(q) \neq 0 \qquad \text{for all } q \in B_\varrho, \quad i = 1, \ldots, d.$$

Now, let $w_1,\dots,w_d\in C_0^\infty(B_\varrho)$ be such that

$$\langle d_q w_i, X_i(q)\rangle = -1 \qquad \text{for all}\ \ q\in B_\varrho, \quad i=1,\dots,d.$$

Then take

$$\mathcal{U}=\bigoplus_{i=1}^{d}\bigl\{a\in C_0^\infty(B_\varrho)\colon\ \|a-\varepsilon w_i\|_1<\delta,\ \|a\|_2<\gamma\bigr\}, \tag{2}$$

where $\delta<\min\bigl\{\frac{\varepsilon}{2},\frac{\varepsilon}{2\|X_1\|_0},\dots,\frac{\varepsilon}{2\|X_d\|_0}\bigr\}$. Let us denote by $U_1,\dots,U_d$ the sets that compose the direct sum (2).

Note that, for every $\gamma>0$, if $\varepsilon<\min\bigl\{\frac{\gamma}{\|w_1\|_2},\dots,\frac{\gamma}{\|w_d\|_2}\bigr\}$, then $\mathcal{U}$ is an open nonempty subset of $C_0^\infty(B_\varrho)^d$.

Let us start with the computation of the differential of F. Set $\phi_i(a_i)=e^{a_iX_i}$ for $i=1,\dots,d$. So

$$F(a)=\phi_1(a_1)\circ\ldots\circ\phi_d(a_d).$$

Now let us compute the differential of ϕ_i for every $i=1,\dots,d$. Since this computation is the same for every i, we omit the subscript. We have

$$\left(\frac{\partial}{\partial a}e^{aX}\right)\colon\ \xi\mapsto\left(\int_0^1 e^{-\int_0^t\langle da,X\rangle\circ e^{\tau aX}\,d\tau}\xi\circ e^{taX}\,dt\right)X\circ e^{aX}.$$

Indeed,

$$\begin{aligned}D\phi(a)[\xi]&=\frac{\partial}{\partial\varepsilon}e^{(a+\varepsilon\xi)X}\Big|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\overrightarrow{\exp}\int_0^1 e^{t\,\mathrm{ad}\,aX}\varepsilon\xi X\,dt\Big|_{\varepsilon=0}\circ e^{aX}\\&=\frac{\partial}{\partial\varepsilon}\overrightarrow{\exp}\int_0^1 e^{taX}\varepsilon\xi e^{t\,\mathrm{ad}\,aX}X\,dt\big|_{\varepsilon=0}\circ e^{aX}\\&=\int_0^1 e^{taX}\xi\,\mathrm{Ad}\,e^{taX}X\,dt\circ e^{aX}.\end{aligned}$$

Now, the time-varying vector field $\mathrm{Ad}\,e^{taX}X$ is the vector field X twisted by the flow of the rescaling by a smooth function a of X itself. We expect that $\mathrm{Ad}\,e^{taX}X$ is a time-dependent rescaling of X. Indeed,

$$\frac{d}{dt}\mathrm{Ad}\,e^{taX}X=\frac{d}{dt}e^{t\,\mathrm{ad}\,aX}X=e^{t\,\mathrm{ad}\,aX}[aX,X]=-e^{taX}\langle da,X\rangle\,\mathrm{Ad}\,e^{taX}X.$$

Then

$$\mathrm{Ad}\,e^{taX}X=e^{-\int_0^t\langle da,X\rangle\circ e^{\tau aX}\,d\tau}X.$$

Let us set

$$A(a)\xi=\int_0^1 e^{-\int_0^t\langle da,X\rangle\circ e^{\tau aX}\,d\tau}\xi\circ e^{taX}\,dt, \tag{3}$$

so

$$D\phi(a)[\xi]=A(a)\xi X\circ\phi(a).$$

Let $a = (a_1, \dots, a_d)$ and $\xi = (\xi_1, \dots, x_d)$; then

$$\begin{aligned} DF(a)[\xi] &= A(a_1)\xi_1 X_1 \circ \phi_1(a_1) \circ \dots \circ \phi_d(a_d) + \phi_1(a_1) \circ A(a_2)\xi_2 X_2 \circ \phi_2(a_2) \circ \dots \circ \phi_d(a_d) \\ &\qquad + \dots + \phi_1(a_1) \circ \dots \circ A(a_d)\xi_d X_d \circ \phi_d(a_d) \\ &= A(a_1)\xi_1 X_1 \circ F(a) + \mathrm{Ad}(\phi_1(a_1))A(a_2)\xi_2 X_2 \circ F(a) \\ &\qquad + \dots + \mathrm{Ad}\big(\phi_1(a_1) \circ \dots \circ \phi_{d-1}(a_{d-1})\big)A(a_d)\xi_d X_d \circ F(a). \end{aligned}$$

We set $u_1 = A(a_1)\xi_1$, $u_2 = \phi_1(a_1)A(a_2)\xi_2$, $\dots$, $u_d = \phi_1(a_1) \circ \dots \circ \phi_{d-1}(a_{d-1})A(a_d)\xi_d$ and $Y_1 = X_1$, $Y_2 = \mathrm{Ad}(\phi_1(a_1)) \circ X_2$, $\dots$, $Y_d = \mathrm{Ad}(\phi_1(a_1) \circ \dots \circ \phi_{d-1}(a_{d-1})) \circ X_d$. Then

$$DF(a)[\xi] = \sum_{i=1}^{d} u_i Y_i \circ F(a).$$

Now we want to solve

$$DF(a)[\xi] = V \tag{4}$$

for every given vector field $V \in \mathrm{Vec}\, B_\varrho$ and $a \in \mathcal{U}$. For γ sufficiently small the vector fields $Y_1, \dots, Y_d$ are linearly independent at 0 and V can be written as $V = \sum_{i=1}^{d} v_i Y_i$. Therefore, in order to find a solution to (4), it is sufficient to solve for every $\eta_i \in C^\infty(B_\varrho)$ the equation

$$A(a_i)\xi_i = \eta_i, \qquad a_i \in U_i, \tag{5}$$

for every $i = 1, \dots, d$.

The next step consists in finding a coordinate system on B_ϱ such that the vector field $a_i X_i$ is linear. Since the argument does not depend on $i = 1, \dots, d$, from now on the subscript i is omitted. For every $a \in U$ we have $a(0) = 0$. Moreover, $\langle d_q a, X(q)\rangle < 0$ for every $q \in B_\varrho$; indeed,

$$\langle d_q a, X(q)\rangle = \langle d_q(a - \varepsilon w), X(q)\rangle + \varepsilon\langle d_q w, X(q)\rangle \le \|a - \varepsilon w\|_1 \|X\|_0 - \varepsilon < -\varepsilon/2.$$

Therefore X is transversal to $a^{-1}(0)$ at every point. In particular, we may rectify the field X in such a way that, in the new coordinates,

$$X = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad a(0, x_2, \dots, x_d) = 0. \tag{6}$$

Moreover, since this system of coordinates depends only on an orthogonality relation with the differential of a, it depends tamely, with degree at most 1, on a.

In order to simplify the notation, we set $x = x_1$ and $y = (x_2, \dots, x_d)$.

The following lemma allows us to consider only the linear part of the field $a\frac{\partial}{\partial x}$. Below we prove only the existence of a change of variables. That it is tame is shown in Subsection 4.1.

Lemma 2.2. *Let $a \in U$. Then there exists a smooth change of coordinates Ψ on B_ϱ that linearizes the vector field $a(x,y)\frac{\partial}{\partial x}$.*

Moreover, Ψ is a tame map with respect to a with tame inverse.

Proof. Since $a(0,y) = 0$, we can write $a(x,y) = -x\alpha(y) + xb(x,y)$, with $b(0,y) = 0$. Consider a solution $x(t)$ of the parametric ODE $\dot{x} = a(x,y)$. We look for a diffeomorphism

$$\Psi(x,y) = (\psi(x,y), y) \tag{7}$$

such that if $z = \psi(x,y)$ then

$$\dot{z} = -\alpha(y)z.$$

Suppose that $\psi(x,y) = x + x\phi(x,y)$, with $\phi(0,y) = 0$. Then

$$\frac{d}{dt}z = \frac{d}{dt}(x + x\phi(x,y)) = \dot{x} + \dot{x}\phi(x,y) + x\dot{x}\frac{\partial\phi}{\partial x}(x,y) = a(x,y)\Big(1 + \phi + x\frac{\partial\phi}{\partial x}(x,y)\Big),$$

and, on the other hand,

$$\frac{d}{dt}z = -\alpha(y)z = -\alpha(y)x(1 + \phi(x,y)).$$

Therefore ϕ is a solution of the following family of ODEs with parameter y:

$$\frac{\partial\phi}{\partial x}(x,y) = -\frac{b(x,y)}{a(x,y)}(1 + \phi(x,y)),$$
$$\phi(0,y) = 0.$$

So

$$\phi(x,y) = e^{-\int_0^x \frac{b(s,y)}{a(s,y)}\,ds} - 1,$$

and

$$\psi(x,y) = xe^{-\int_0^x \frac{b(s,y)}{a(s,y)}\,ds}. \tag{8}$$

We have an explicit formula for the change of coordinates Ψ. $\square$

Lemma 2.2 allows us to assume that $aX = -\alpha(y)x\frac{\partial}{\partial x}$, where $-\alpha(y) = \frac{\partial a}{\partial x}(0,y)$. Hence

$$\xi \circ e^{taX}(x,y) = \xi\big(e^{-t\alpha(y)}x, y\big),$$

which implies

$$\int_0^1 e^{\int_0^t e^{-\tau\alpha(y)x\frac{\partial}{\partial x}}\alpha(y)\,d\tau}\xi\big(e^{-t\alpha(y)}x,y\big)\,dt = \int_0^1 e^{t\alpha(y)}\xi\big(e^{-t\alpha(y)}x,y\big)\,dt. \tag{9}$$

Set

$$\widehat{A}(\xi) = \int_0^1 e^{t\alpha(y)}\xi\big(e^{-t\alpha(y)}x,y\big)\,dt. \tag{10}$$

In this last step we want to prove that this map $\widehat{A}$ has a smooth family of right inverses. Moreover, in Section 4 we prove that $\widehat{A}$ is a tame map (see Subsection 4.2) with a tame family of right inverses (Subsection 4.3).

Let

$$\xi(x,y) = \xi(0,y) + x\xi_x(0,y) + x^2u(x,y).$$

Then

$$\widehat{A}(\xi(0,y)) = \frac{e^{\alpha}(y) - 1}{\alpha(y)}\xi(0,y)$$

and

$$\widehat{A}(x\xi_x(0,y)) = x\xi_x(0,y).$$

Now let

$$v(x,y)=\frac{1}{x}\int\limits_0^x u(s,y)\,ds$$

and

$$R\colon\ v(x,y)\mapsto e^{-\alpha(y)}v\big(e^{-\alpha(y)}x,y\big).$$

Then

$$\begin{aligned}\widehat{A}(x^2u(x,y))&=x^2\int\limits_0^1 e^{-\alpha(y)t}u\big(e^{-t\alpha(y)}x,y\big)\,dt=\frac{x^2}{\alpha(y)}\int\limits_{e^{-\alpha(y)}}^1 u(\tau x,y)\,d\tau\\&=\frac{x^2}{\alpha(y)}\big(v(x,y)-e^{-\alpha(y)}v\big(e^{-\alpha(y)}x,y\big)\big)\\&=\frac{x^2}{\alpha(y)}(I-R)v(x,y).\end{aligned}$$

Let $\|v\|_{C^{n,0}}=\sup_{1\le i\le n}\big\|\frac{\partial^i v}{\partial x^i}\big\|_{C^0}$. Since $\alpha(y)$ is uniformly bounded away from 0, it is clear that R is a contraction from the space $C^{\infty,0}$ of continuous functions smooth with respect to x into itself. Hence $I-R$ is invertible in this space and $(I-R)^{-1}=\sum_{k=0}^{\infty}R^k$ maps a function f, smooth on the box, in a continuous function $g=(I-R)^{-1}f$ smooth with respect to x. We want to prove that if $f\in C^\infty$ then $g=(I-R)^{-1}f\in C^\infty$. Let us do this by induction. Suppose that $g\in C^{\infty,0}$. Then

$$D_yg=(I-R)^{-1}\big(f_y+\alpha'e^{-\alpha}g+\alpha'e^{-2\alpha}g_x\big)\in C^{\infty,0},$$

so $g\in C^{\infty,1}$.

Now let $n\ge1$ and suppose that $g\in C^{\infty,n-1}$. Then

$$\begin{aligned}D_y^nf(x,y)&=D_y^n(I-R)g=D_y^ng(x,y)-D_y^n\big(e^{-\alpha(y)}g\big(e^{-\alpha(y)}x,y\big)\big)\\&=D_y^ng(x,y)+\sum_{k=0}^{n}\binom{n}{k}D_y^ke^{-\alpha(y)}D_y^{n-k}g\big(e^{-\alpha(y)}x,y\big)\\&=(I-R)D_y^ng(x,y)+\sum_{k=1}^{n}\binom{n}{k}D_y^ke^{-\alpha(y)}D_y^{n-k}g\big(e^{-\alpha(y)}x,y\big),\end{aligned}$$

so we have

$$D_y^ng(x,y)=(I-R)^{-1}\left(D_y^nf(x,y)-\sum_{k=1}^{n}\binom{n}{k}D_y^ke^{-\alpha(y)}D^{n-k}g\big(e^{-\alpha(y)}x,y\big)\right),$$

which is $C^{\infty,0}$ by hypothesis. Therefore $g\in C^{\infty,n}$.

Therefore $\widehat{A}$ is invertible and we have an explicit formula for the inverse of $\widehat{A}$. By

$$\widehat{A}(\xi)=\eta,$$

we get

$$u(x,y)=\frac{\partial}{\partial x}\left(x(I-R)^{-1}\frac{\alpha(y)}{x^2}(\eta(x,y)-\eta(0,y))\right)=\frac{\alpha(y)}{x^2}\sum_{k=0}^{\infty}G_k\big(e^{-\alpha(y)k}x,y\big),\qquad(11)$$

where

$$G_k(x,y) := e^{\alpha(y)k}\big(x\eta_x(x,y) - \eta(x,y) + \eta(0,y)\big).$$

Then, finally,

$$\widehat{A}^{-1}(\eta) = \frac{\alpha(y)}{e^{\alpha}(y) - 1}\eta(0,y) + x\eta_x(0,y) + \alpha(y)\sum_{k=0}^{\infty} G_k\big(e^{-\alpha(y)k}x, y\big). \tag{12}$$

This completes the proof of Lemma 2.1. $\square$

3. PROOF OF THE THEOREM

Let, for $q \in M$,

$$\mathcal{P} = \mathrm{Gr}\{af\colon\ a \in C^\infty(M),\ f \in \mathcal{F}\} \qquad \text{and} \qquad \mathcal{P}_q = \{P \in \mathcal{P}\colon\ P(q) = q\}.$$

Lemma 3.1. *Any $q \in M$ possesses a neighborhood $U_q \subset M$ such that the set*

$$\big\{P|_{U_q}\colon\ P \in \mathcal{P}_q\big\} \tag{13}$$

contains a neighborhood of the identity in $C^\infty_q(U_q, M)$.

Proof. According to the orbit theorem of Sussmann [7] (see also the textbook [1]), the transitivity of the action of $\mathrm{Gr}\,\mathcal{F}$ on M implies that

$$T_qM = \mathrm{span}\big\{P_* f(q)\colon\ P \in \mathrm{Gr}\,\mathcal{F},\ f \in \mathcal{F}\big\}.$$

Take $X_i = P_{i*}f_i$, $i = 1,\dots,d$, with $P_i \in \mathrm{Gr}\,\mathcal{F}$ and $f_i \in \mathcal{F}$ in such a way that $X_1(q),\dots,X_d(q)$ form a basis of T_qM. Then, for all smooth functions $a_1,\dots,a_d$ vanishing at q, the diffeomorphism

$$e^{a_1X_1} \circ \ldots \circ e^{a_dX_d} = P_1 \circ e^{(a_1\circ P_1)f_1} \circ P_1^{-1} \circ \ldots \circ P_d \circ e^{(a_d\circ P_d)f_d} \circ P_d^{-1}$$

belongs to the group $\mathcal{P}_q$. By the main lemma the set (13) contains an open subset of $C^\infty_q(U_q, M)$, say $\mathcal{A}$. Now consider $P_0|_{U_q} \in \mathcal{A}$; then $P_0^{-1} \circ \mathcal{A}$ is a neighborhood of the identity contained in (13). $\square$

Definition 1. Given $P \in \mathrm{Diff}(M)$, we set $\mathrm{supp}\,P = \overline{\{x \in M\colon P(x) \neq x\}}$.

Lemma 3.2. *Let $\mathcal{O}$ be a neighborhood of the identity in* $\mathrm{Diff}(M)$*. Then, for any $q \in M$ and any neighborhood $U_q \subset M$ of q, we have*

$$q \in \mathrm{int}\big\{P(q)\colon\ P \in \mathcal{O} \cap \mathcal{P},\ \mathrm{supp}\,P \subset U_q\big\}.$$

Proof. Consider d vector fields $X_1,\dots,X_d$ as in the proof of Lemma 3.1 and let $b \in C^\infty(M)$ be a cut-off function such that $\mathrm{supp}\,b \subset U_q$ and $q \in \mathrm{int}\,b^{-1}(1)$. Then the diffeomorphism

$$Q(s_1,\dots,s_d) = e^{s_1bX_1} \circ \ldots \circ e^{s_dbX_d}$$

belongs to $\mathcal{O} \cap \mathcal{P}$ for any d-tuple of real numbers $(s_1,\dots,s_d)$ sufficiently close to 0. Moreover, $\mathrm{supp}\,Q(s_1,\dots,s_d) \subset U_q$. On the other hand, the map

$$(s_1,\dots,s_d) \mapsto Q(s_1,\dots,s_d)(q)$$

is a local diffeomorphism in a neighborhood of 0. $\square$

The next lemma is due to Palis and Smale (see Lemma 3.1 in [6]).

Lemma 3.3. *Let $\bigcup_j U_j = M$ be a covering of M by open subsets and $\mathcal{O}$ be a neighborhood of the identity in* $\mathrm{Diff}(M)$*. Then the group* $\mathrm{Diff}_0(M)$ *is generated by the subset* $\{P \in \mathcal{O}\colon$ *$\exists j$ such that* $\mathrm{supp}\,P \subset U_j\}$.

Proof of the theorem. According to Lemma 3.3, it is sufficient to prove that, for every $q \in M$, there exist a neighborhood $U_q \subset M$ and a neighborhood of the identity $\mathcal{O} \subset \mathrm{Diff}(M)$ such that any diffeomorphism $P \in \mathcal{O}$ whose support is contained in U_q belongs to $\mathcal{P}$. Moreover, Lemma 3.2 allows us to assume that $P(q) = q$. Finally, Lemma 3.1 completes the proof. $\square$

4. TAME ESTIMATES

To prove tame estimates for the maps involved in the proof of Lemma 2.1, we first need some preliminaries about the C^∞ topology and tame maps. In this section we give only some statement: for proofs, remarks and examples we refer to the paper by Hamilton and, in particular, to [3, Sect. II.2].

Let B be an open subset of $\mathbb{R}^d$. We consider coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ on B. For $f \in C^\infty(B)$ we denote by $D^n f$ the nth differential of f. Moreover, we set

$$\frac{\partial^n}{\partial x^n} f := D^n_x f = f^{(n,0)} \qquad \text{and} \qquad \frac{\partial^n}{\partial y^n} f := D^n_y f = f^{(0,n)}.$$

We will choose at every line the more efficient and brief notation from the two above. Note that, in order to simplify the notation, we denote the differential of a function with respect to y treating y as a 1-dimensional variable.

Finally, the letter C denotes a strictly positive constant, whose value may change from line to line.

Definition 2. Let X and Y be tame spaces and $T\colon U \subset X \to Y$. We say that T satisfies *tame estimates* of degree r and base n_0 if

$$\|T(f)\|_n \le C(\|f\|_{n+r} + 1)$$

for all $f \in U$ and all $n \ge n_0$. We say that T is a (*smooth*) *tame map* if T is smooth and all its derivatives satisfy tame estimates. Note that the constant C may depend on n and we allow r, n_0, and C to vary from neighborhood to neighborhood.

A useful property of tame maps is the following

Proposition 4.1. *A composition of tame maps is tame.*

A remarkable property of smooth maps is the so-called "interpolation inequality":

Proposition 4.2 (interpolation inequalities)**.** *Let M be a compact manifold and $\ell \le m \le n$. Then, for all $f \in C^\infty(M)$, we have*

$$\|f\|_m^{n-\ell} \le C\|f\|_n^{m-\ell}\|f\|_\ell^{n-m}. \tag{14}$$

For a pair of functions we have the following corollary.

Corollary 4.1. *Let $f, g \in C^\infty(M)$. If (i, j) lies on the segment joining (k, ℓ) and (m, n), then there exists a constant C independent of f and g such that*

$$\|f\|_i\|g\|_j \le C\bigl(\|f\|_k\|g\|_\ell + \|f\|_m\|g\|_n\bigr).$$

A tool used in this paper is the iterated chain rule formula also known as "Faá di Bruno's formula" (see, for example, [5]). Let f and g be smooth functions from U and $V \subset \mathbb{R}^d$, respectively, to $\mathbb{R}^d$ such that $U \subset g(V)$. Then

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{m_1!\, m_2!\, \ldots\, m_n!} f^{(m_1+\ldots+m_n)}(g(x)) \prod_{j=1}^{n} \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j}, \tag{15}$$

where the sum is over all the n-tuples $(m_1, \ldots, m_n)$ such that $m_1 + 2m_2 + \ldots + nm_n = n$.

As a consequence of this formula and of the interpolation inequalities, we have the following proposition.

Proposition 4.3 (composition is tame). *If there exists K such that $\|f\|_1 \le K$ and $\|g\|_1 \le K$, then*

$$\|f \circ g\|_n \le C(\|f\|_n + \|g\|_n + 1). \tag{16}$$

From “Faá di Bruno’s formula” it directly follows that

$$\sup_x |D^n f^k(x)| \le C\|f\|_n \|f\|_0^{k-1}.$$

And if $f \in C^\infty(U)$ is such that there exist $0 < \alpha < \beta$ such that $\alpha < f(x) < \beta$ for all $x \in U$, then

$$\sup_x \left| D^n \frac{1}{f(x)} \right| \le C\|f\|_n, \tag{17}$$

where C depends on n, α, β, and U.

An example of a tame map is given by the exponential map. Indeed, the following proposition holds true.

Proposition 4.4. *Let $U \subset \mathbb{R}^d$. Then the map $f \in C^\infty(U) \mapsto e^f \in C^\infty(U)$ is tame of degree 0.*

This proposition can be easily generalized to the exponential map from $\operatorname{Vec} U$ to $\operatorname{Diff}_0(U)$.

Corollary 4.2. *Let $U \subset \mathbb{R}^d$. Then the exponential map* $\exp\colon \operatorname{Vec} U \to \operatorname{Diff}(U)$ *is tame.*

4.1. Tame estimates for the change of coordinates. Here we prove that the change of coordinates Ψ defined in (7) and (8) is a tame map. Clearly, it is sufficient to find tame estimates for

$$\psi = xe^{-\int_0^x \frac{b(s,y)}{a(s,y)}\,ds}.$$

Since the exponential is tame of degree 0, we have

$$\|\psi\|'_n := \left\|\frac{\psi}{x}\right\|_n \le C \left\| \int_0^x \frac{b(s,y)}{a(s,y)}\,ds \right\|_n,$$

where

$$\|f\|'_n := \|f/x\|_n$$

is a tamely equivalent grading on $C_0^\infty(B_\varrho)$.

Now, consider the nth derivative with respect to y, that is,

$$\left| D_y^n \int_0^x \frac{b(s,y)}{a(s,y)}\,ds \right| = \left| D_y^n \int_0^x \frac{b_x(\nu(s),y)}{-\alpha(y)+b(s,y)}\,ds \right| \le C\biggl(\|b_x\|_n \left\| \frac{1}{-\alpha+b} \right\|_0 + \|b_x\|_0 \left\| \frac{1}{-\alpha+b} \right\|_n \biggr).$$

Since

$$-\alpha(y) + b(x,y) + \varepsilon \le \|a - \varepsilon w\|_1 < \varepsilon/2,$$

it follows that $-\alpha(y) + b(x,y) < -\varepsilon/2$ and so $|-\alpha(y) + b(x,y)|$ is uniformly bounded away from 0. Therefore, applying property (17), we find

$$\sup \left| D_y^n \int_0^x \frac{b(s,y)}{a(s,y)}\,ds \right| \le C\bigl(\|b\|_{n+1} + \|-\alpha + b\|_n\bigr) \le C\bigl(\|a\|'_{n+1} + 1\bigr).$$

The remaining terms are of the form $D_y^{n-k}D_x^k$ with $k\ge 1$. So

$$\left|D_y^{n-k}D_x^k\int\limits_0^x \frac{b(s,y)}{a(s,y)}\,ds\right| = \left|D_y^{n-k}D_x^{k-1}\frac{b(x,y)}{a(x,y)}\right| \le \left\|\frac{b(x,y)}{a(x,y)}\right\|_{n-1}$$
$$\le C\biggl(\left\|\frac{b(x,y)}{x}\right\|_{n-1} + \left\|\frac{1}{-\alpha+b}\right\|_{n-1}\biggr). \tag{18}$$

Now, by Taylor's expansion of b in x, we obtain

$$D_x^q\frac{b(x,y)}{x} = (-1)^q\frac{q!}{x^q}\sum_{\ell=0}^{q}(-1)^\ell\,\frac{b^{(q+1,0)}(\nu_\ell(x),y)x^{q+1}}{(q-\ell+1)!\,\ell!},$$

so that

$$\sup\left|D_y^pD_x^q\frac{b(x,y)}{x}\right| \le C\sup\bigl|b^{(q+1,p)}(x,y)\bigr|.$$

Hence

$$\left\|\frac{b(x,y)}{x}\right\|_{n-1} \le C\|b\|_n.$$

Therefore by (18) and (17) we get the tame estimates

$$\left|D_y^{n-k}D_x^k\int\limits_0^x\frac{b(s,y)}{a(s,y)}\,ds\right| \le C\bigl(\|a\|_n'+1\bigr),$$

and we prove that Ψ is a tame map of degree 1.

Now, let us prove that the inverse is tame too. Let $g(x,y)$ be a smooth function such that

$$\psi(g(x,y),y) = x. \tag{19}$$

It is sufficient to prove tame estimates for g. First of all, note that

$$|\psi_x(x,y)| = \frac{\alpha(y)}{\alpha(y)-b(x,y)}\,e^{-\int_0^x \frac{b(s,y)}{a(s,y)}\,ds} \ge \frac{\varepsilon-\delta}{\varepsilon+\delta}\,e^{-\varrho\frac{\delta}{\varepsilon-\delta}}$$

for every $(x,y)\in B_\varrho$. Hence

$$\|1/\psi_x\|_0 \le C_1, \tag{20}$$

where C_1 depends only on ε and ϱ. Differentiating (19) with respect to x, we have

$$g_x(x,y) = \frac{1}{\psi_x(g(x,y),y)}.$$

Then we have an upper bound for g:

$$|g(x,y)| \le |x|\sup|g_x(x,y)| \le \varrho C_1 =: C_2.$$

Moreover, $\|\psi\|_1 \le C\|a\|_2 \le C_3$.

Now let $n>1$. By the iterated chain rule (15) we have

$$D^n\psi(g(x,y),y) = \psi_x(g(x,y),y)D^n(g(x,y),y) + \sum_{\Pi} c_kD^k\psi\big|_{(g(x,y),y)}\prod_{j=1}^{k-1}\bigl(D^j(g(x,y),y)\bigr)^{m_j},$$

where the sum is over the set Π of all $(n-1)$-tuples $(m_1,\dots,m_{n-1})$ with $m_1+\ldots+(n-1)m_{n-1}=n$ and where we denote for simplicity $k=m_1+\ldots+m_{n-1}$.

On the other hand, differentiating (19) n times, we have

$$D^n\psi(g(x,y),y)=0.$$

Therefore

$$\begin{aligned}\|(g(x,y),y)\|_n &\le C\|1/\psi_x\|_0\left\|\sum D^k\psi\big|_{(g(x,y),y)}\prod_{j=1}^{k-1}\big(D^j(g(x,y),y)\big)^{m_j}\right\|_0\\ &\le C\sum_{\Pi}\|\psi\|_k\|g\|_1^{m_1}\ldots\|g\|_{n-1}^{m_{n-1}}.\end{aligned}$$

By interpolation inequalities (14) we have, for every $j=1,\dots,n-1$,

$$\|g\|_j^{m_j}\le C\|g\|_0^{\frac{n-j-1}{n-1}m_j}\|g\|_{n-1}^{\frac{j-1}{n-1}m_j}$$

and also

$$\|\psi\|_k\le C\|\psi\|_1^{\frac{n-k}{n-1}}\|\psi\|_n^{\frac{k-1}{n-1}}.$$

Hence,

$$\begin{aligned}\|\psi\|_k\|g\|_1^{m_1}\ldots\|g\|_{n-1}^{m_{n-1}} &\le C\left(\|g\|_0^{\frac{nk-n-1}{n-1}}\|g\|_{n-1}^{\frac{n-k}{n-1}}\|\psi\|_1^{\frac{n-k}{n-1}}\|\psi\|_n^{\frac{k-1}{n-1}}\right)\\ &\le C\|g\|_0^{\frac{n(k-1)-k}{n-1}}\big(\|g\|_0\|\psi\|_n\big)^{\frac{k-1}{n-1}}\big(\|g\|_{n-1}\|\psi\|_1\big)^{\frac{n-k}{n-1}}\\ &\le C\|g\|_0^{\frac{n(k-1)-k}{n-1}}\big(\|g\|_0\|\psi\|_n+\|g\|_{n-1}\|\psi\|_1\big).\end{aligned}$$

Then, by the bounds on $\|g\|_0$ and $\|\psi\|_1$, there exists a constant C, depending only on ϱ,γ, and ε, such that

$$\|g\|_n\le C\big(\|\psi\|_n+\|g\|_{n-1}\big).$$

By induction and using the fact that ψ is tame, we find that g is a tame map and so is the inverse of ψ.

4.2. Tame estimates for $\widehat{A}$. Here we give tame estimates for

$$\widehat{A}(\xi)=\int_0^1 e^{t\alpha(y)}\xi\big(e^{-t\alpha(y)}x,y\big)\,dt.$$

We have

$$\big|D^n\widehat{A}(\xi(x,y))\big|=\left|D^n\int_0^1 e^{t\alpha(y)}\xi\big(e^{-t\alpha(y)}x,y\big)\,dt\right|\le C\int_0^1\sum_{j=0}^n\big|D^j\xi\big(e^{-t\alpha(y)}x,y\big)\big|\cdot\big|D^{n-j}e^{t\alpha(y)}\big|.$$

Set $h(x,y)=(e^{-\alpha(y)t},y)$. Since the exponential is tame (Proposition 4.4), we have $\|h\|_j\le C\|\alpha\|_j$ for every j. In particular, $\|h\|_1$ is bounded by a constant independent of α. Therefore, by (16), on every open subset of $C^\infty(B_\varrho)$ of the form $\|\xi\|_0\le K$, we obtain

$$\|\widehat{A}(\xi)\|_n\le C\sum_{j=0}^n\big(\|\xi\|_j+\|h\|_j\|\xi\|_1\big)\|\alpha\|_{n-j}\le C\big(\|\xi\|_n+\|\alpha\|_n\|\xi\|_1\big),$$

and $\widehat{A}$ is a tame linear map of degree 0.

4.3. Tame estimates for $\widehat{A}^{-1}$. In order to prove that $\widehat{A}^{-1}$ is tame, let us verify tame estimates for the first term of (12). Note that

$$\left\|\frac{\alpha(y)}{e^{\alpha(y)}-1}\right\|_0 \le C.$$

Moreover, by inequality (17),

$$\left\|\frac{1}{e^{\alpha(y)}-1}\right\|_n \le \|e^{\alpha(y)}-1\|_n \le \|\alpha\|_n;$$

therefore

$$\left\|\frac{\alpha(y)}{e^{\alpha(y)}-1}\right\|_n \le \|\alpha\|_n.$$

Finally, the first term is tame since

$$\left\|\frac{\alpha(y)}{e^{\alpha}(y)-1}\eta(0,y)\right\|_n \le C\big(\|\alpha\|_n + \|\eta\|_n\big).$$

For the second term, we have

$$\|x\eta_x(0,y)\|_n = \sup_{1\le j\le n}\big\|x\eta^{(1,j)}(0,y)+\eta^{(1,j-1)}(0,y)\big\|_0 \le C\|\eta\|_{n+1}.$$

Now consider the last term. Note that, using the Lagrange formula twice, we have

$$\big\|G_k\big(e^{-\alpha(y)k}x,y\big)\big\|_0 = \big\|x\big(\eta_x\big(e^{-\alpha(y)k}x,y\big)-\eta_x(\nu_k(x,y),y)\big)\big\|_0 \le \big\|x^2e^{-\alpha(y)k}\big\|_0\|\eta_{xx}\|_0;$$

therefore,

$$\left\|\sum_{k=0}^{\infty}G_k\big(e^{-\alpha(y)k}x,y\big)\right\|_0 \le C\|\eta\|_2.$$

Now,

$$\left\|\alpha(y)\sum_{k=0}^{\infty}G_k\big(e^{-\alpha(y)k}x,y\big)\right\|_n \le C\left(\|\alpha\|_n\|\eta\|_2 + \left\|\sum_{k=0}^{\infty}G_k\big(e^{-\alpha(y)k}x,y\big)\right\|_n\|\alpha\|_0\right).$$

Then it remains to estimate only the quantity

$$\left\|\sum_{k=0}^{\infty}G_k\big(e^{-\alpha(y)k}x,y\big)\right\|_n.$$

For all n and k we have

$$\big|D^ne^{\alpha(y)k}\big| \le C\|\alpha\|_nk^ne^{\alpha(y)k} \qquad \text{and} \qquad \big|D^ne^{-\alpha(y)k}\big| \le C\|\alpha\|_nk^ne^{-\alpha(y)k}. \tag{21}$$

Moreover, the following estimates hold:

$$\begin{aligned}\big|D_y^rG_k(x,y)\big| &\le \sum_{p=0}^{r}\binom{r}{p}\big|D_y^{r-p}e^{\alpha(y)k}\big|\cdot\big|D_y^p\big(x\eta_x(x,y)-\eta(x,y)+\eta(0,y)\big)\big|\\ &\le C\sum_{p=0}^{r}\|\alpha\|_{r-p}k^{r-p}e^{\alpha(y)k}\big|x\eta^{(1,p)}(x,y)-\eta^{(0,p)}(x,y)+\eta^{(0,p)}(0,y)\big|\\ &\le C\sum_{p=0}^{r}\|\alpha\|_{r-p}k^{r-p}e^{\alpha(y)k}\big|x\big(\eta^{(1,p)}(x,y)-\eta^{(1,p)}(\nu_k(x,y),y)\big)\big|\\ &\le Ck^re^{\alpha(y)k}|x|\big(\|\alpha\|_{r+1}\|\eta\|_0+\|\alpha\|_0\|\eta\|_{r+1}\big),\end{aligned} \tag{22}$$

and therefore

$$\left|D_y^r G_k(x,y)\right|\big|_{(e^{-\alpha(y)k}x,y)} \le Ck^r\big(\|\alpha\|_{r+1}+\|\eta\|_{r+1}\big).$$

Now, let $j\ge 1$ and consider

$$\begin{aligned}\left|D_y^{r-j}D_x^j G_k(x,y)\right| &= \left|D_y^{r-j}\,e^{\alpha(y)k}D_x^{j-1}(x\eta_{xx}(x,y))\right|\\ &= \left|D_y^{r-j}e^{\alpha(y)k}\big(x\eta^{(j+1,0)}(x,y)+(j-1)\eta^{(j,0)}(x,y)\big)\right|\\ &\le C\sum_{p+q=r-j}\left|D_y^q e^{\alpha(y)k}\right|\cdot\left|x\eta^{(j+1,p)}(x,y)+(j-1)\eta^{(j,p)}(x,y)\right|\\ &\le Ck^{r-j}\big(1+(j-1)e^{\alpha(y)k}\big)\big(\|\alpha\|_{r+1}\|\eta\|_0+\|\alpha\|_0\|\eta\|_{r+1}\big).\end{aligned}\tag{23}$$

By (15), if $m_1+\ldots+nm_n=n$ and $r=m_1+m_2+\ldots+m_n$, then

$$\left|D^nG_k\big(e^{-\alpha(y)k}x,y\big)\right|\le\sum_{\Pi}\left|D_y^rG_k(x,y)\right|_{(e^{-\alpha(y)k}x,y)}\Big|\prod_{j=1}^n\left|D^j\big(e^{-\alpha(y)k},y\big)\right|^{m_j}$$

and, by (22) and (23),

$$\left|D^nG_k\big(e^{-\alpha(y)k}x,y\big)\right|\le\sum_{\Pi}Ck^re^{-\alpha(y)kr}\big(\|\alpha\|_{r+1}\|\eta\|_0+\|\eta\|_{r+1}\|\alpha\|_0\big)\prod_{j=1}^n\|\alpha\|_j^{m_j}.$$

Interpolation inequalities (14) imply

$$\|\alpha\|_{r+1}\le C\|\alpha\|_1^{\frac{n-r}{n}}\|\alpha\|_{n+1}^{\frac{r}{n}},\tag{24}$$

and, for $j=1,\ldots,n$,

$$\|\alpha\|_j\le C\|\alpha\|_1^{\frac{n+1-j}{n}}\|\alpha\|_{n+1}^{\frac{j-1}{n}},\tag{25}$$

since $\|\alpha\|_1$ is bounded and if $\|\eta\|_0\le C$, then

$$\|\alpha\|_{r+1}\|\eta\|_0\|\alpha\|_1^{m_1}\ldots\|\alpha\|_n^{m_n}\le C\|\alpha\|_{n+1}.$$

In a similar way we can prove

$$\|\eta\|_{r+1}\le C\|\eta\|_1^{\frac{n-r}{n}}\|\eta\|_{n+1}^{\frac{r}{n}},$$

which implies, together with (25), that

$$\|\eta\|_{r+1}\|\alpha\|_0\|\alpha\|_1^{m_1}\ldots\|\alpha\|_n^{m_n}\le C(\|\eta\|_{n+1})^{\frac{r}{n}}(\|\eta\|_1\|\alpha\|_{n+1})^{\frac{n-r}{n}}\le C\big(\|\eta\|_{n+1}+\|\eta\|_1\|\alpha\|_{n+1}\big).$$

Therefore, finally,

$$\begin{aligned}\left\|\sum_{k=0}^\infty G_k\big(e^{-\alpha(y)k}x,y\big)\right\|_n &\le\sum_{k=0}^\infty\left\|G_k\big(e^{-\alpha(y)k}x,y\big)\right\|_n\\ &\le\sum_{k=0}^\infty\left\|\sum D_y^rG_k(x,y)\right|_{(e^{-\alpha k}x,y)}\prod_{j=1}^n\big(D^j\big(e^{-\alpha(y)k},y\big)\big)^{m_j}\Big\|_0\\ &\le C\sum_{k=0}^\infty k^n\left\|e^{-\alpha(y)k}\right\|_0\big(\|\eta\|_{n+1}+\|\alpha\|_{n+1}\|\eta\|_1\big)\\ &= C\big(\|\eta\|_{n+1}+\|\alpha\|_{n+1}\|\eta\|_1\big).\end{aligned}$$

This completes the proof of the fact that $\widehat{A}^{-1}$ is a tame map.

REFERENCES

1. A. A. Agrachev and Yu. L. Sachkov, *Control Theory from the Geometric Viewpoint* (Springer, Berlin, 2004).
2. W.-L. Chow, "Über Systeme von linearen partiellen Differentialgleichungen erster Ordinung," Math. Ann. **117**, 98–105 (1939).
3. R. S. Hamilton, "The Inverse Function Theorem of Nash and Moser," Bull. Am. Math. Soc. **7**, 65–222 (1982).
4. C. Lobry, "Une propriété générique des couples de champs de vecteurs," Czech. Math. J. **22**, 230–237 (1972).
5. S. Roman, "The Formula of Faá di Bruno," Am. Math. Mon. **87** (10), 805–809 (1980).
6. J. Palis and S. Smale, "Structural Stability Theorems," in *Global Analysis, Berkeley, Calif., 1968* (Amer. Math. Soc., Providence, RI, 1970), Proc. Symp. Pure Math. **14**, pp. 223–231.
7. H. J. Sussmann, "Orbits of Families of Vector Fields and Integrability of Distributions," Trans. Am. Math. Soc. **180**, 171–188 (1973).
8. P. K. Rashevskii, "On Connecting Any Two Points of a Completely Nonholonomic Space by an Admissible Curve," Uch. Zap. Mosk. Ped. Inst. im. Libknekhta, Ser. Fiz.-Mat. Nauk, No. 2, 83–94 (1938).
9. W. Thurston, "Foliations and Groups of Diffeomorphisms," Bull. Am. Math. Soc. **80**, 304–307 (1974).

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