On the control of the Heider balance model*

S. Wongkaew† M. Caponigro ‡ K. Kułakowski§ A. Borzì¶

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Abstract

The Heider social balance model describes the evolution of the relationships in a social network of humans or animals. This model is built upon the concept of balance of triads consisting of friendly or hostile edges representing the state of the network. In this differential model, a leader is introduced in order to control the system and to drive the social network to a desired relationship state. Further, the stability, the local controllability, and the optimal control through leadership of the Heider model are investigated. Results of numerical experiments demonstrate the ability of the proposed control strategy to drive the Heider balance model to friendship.

1 Introduction

The balance theory proposed by F. Heider [13, 14] attempts to model how individuals develop their relationships with other individuals and with objects in their environment based on a cognitive consistency motive that drives toward psychological balance. This motive urges to maintain one's values and beliefs over time resulting in the preference to have a balanced state where the affect valence in the system multiplies out to a positive result. Specifically, in the relation of three individuals, balance state occurs when all sign multiplication of sentiment relations is positive. In this way,

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†Institut für Mathematik, Universität Würzburg, Emil-Fischer-Strasse 31, 97074 Würzburg, Germany (suttida.wongkaew@mathematik.uni-wuerzburg.de).

‡Conservatoire National des Atrs et Métiers, Équipe M2N, 292 rue Saint-Martin, 75003, Paris, France (marco.caponigro@cnam.fr)

§Faculty of Physics and Applied Computer Science, AGH University of Science and Technology al. Mickiewicza 30, PL-30059 Krakòw, Poland (kulakowski@novell.ftj.agh.edu.pl).

¶Institut für Mathematik, Universität Würzburg, Emil-Fischer-Strasse 30, 97074 Würzburg, Germany (alfio.borzi@mathematik.uni-wuerzburg.de).

the balance state occurs when the sentiment relations are all positive or two negatives and one positive. We refer to sentiment relation between two people as a linking edge to which a positive value is associated in case of friendship or a negative value in case of hostility. We remark that Heider balance has been identified experimentally in society of hyraxes [15] and, in particular, this model can be applied to certain mammals; see also [18] for further discussion.

In a system of many people or animals, the concept of social balance is related to the balance of each triad consisting of friendly or hostile edges. The resulting system can be investigated in the framework of network dynamics by using mathematical modeling based on agent-based simulation and in the framework of graph theory where nodes represent individuals and their links represent relationships; see [1, 2, 3, 7, 17, 19, 20, 22, 23] for a partial list of references on these approaches.

From a mathematical point of view, it is certainly advantageous to consider a continuous time Heider balance system [16]. Indeed, in this case powerful tools for the investigation of the dynamics of this system can be applied; we refer to [16, 17] for some fundamental results and to [2, 3, 17, 22] for further developments and applications.

Our purpose is to investigate an optimal control strategy for the continuous time Heider balance (HB) model proposed in [16]. In this strategy, an additional 'reference' agent enters in the network with partial or full connection to the other individuals of the network. This agent acts on the network by modifying the values of the edges that connect to it with the purpose to attain a desired objective. The motivation for this approach is twofold. First, it is more realistic as it describes real social behavior as, for example, in politics. Second, it can be implemented as soon as a reference agent is available. For a similar control method, we refer to [5, 24], however in the present case, the control functions represent the values of the edges connecting to the reference agent, while in [5, 24] the control is implemented in the leader agent.

Our work is organized as follows. In Section 2, we begin with a survey on the continuous time HB model. In addition, the issue of stability is explored. Section 3 is devoted to the local controllability properties of the HB model where a leader is added to the network and the control input is implemented on the links between the leader and the other individuals. In Section 4, we formulate an optimal control problem governed by the HB model with an objective function of the final observation and of the cost of the control. Correspondingly, we introduce the optimality system representing the first-order optimality conditions. We also illustrate a Runge-Kutta discretization scheme that guarantees high-order accuracy of the numerical solution of the optimal control problem. In Section 5, results of numerical experiments demonstrate the validity of our control strategy to drive the Heider balance model to friendship. A section of conclusion completes this work.

2 The continuous time Heider balance model

The structure of the Heider balance model consists in a signed graph where its nodes represent individuals and the corresponding valued pairwise links denote the relationships. The continuous time HB model involving $N > 2$ agents is proposed in [16]. It is given by

$$\dot{x}_{ij}(t) \quad = \quad c(x_{ij}(t); R) \sum_{\substack{k=1 \\ k \neq i,j}}^{N} x_{ik}(t) x_{kj}(t), \quad \text{for} \quad i, j = 1, ..., N, i \neq j, \quad (1)$$

with given initial conditions $x_{ij}(0) = x_{ij}^0$. The indices i, j represent the individuals in the network while $x_{ij} \in \mathbb{R}$ denotes the relationship between agents i and j. A positive value of x_{ij} determines friendship; conversely, a negative value of x_{ij} expresses hostility. Namely

$$\text{sign}(x_{ij}) \quad := \quad \begin{cases} 1, & \text{if} \quad i \text{ and } j \text{ are friends,} \\ 0, & \text{if} \quad i \text{ and } j \text{ have no relationship,} \\ -1, & \text{if} \quad i \text{ and } j \text{ are enemies.} \end{cases} \quad (2)$$

In the dynamics given by (1), we assume that $x_{ii} = 0$, that each agent is connected to all agents in the network, that is, social structure can be seen as fully connected graph, and that $x_{ij} = x_{ji}$ for any i and j.
The function $c : \mathbb{R} \to \mathbb{R}$ in (1) is defined by

$$c(x_{ij}; R) = \frac{1}{N-2} \left(1 - \frac{x_{ij}^2}{R^2} \right), \quad R > 0. \quad (3)$$

The structure of the function $c(x_{ij}; R) := c_{ij}$ is not unique and it is added to system for the purpose of a well-behaved evolution, in the sense to keep the relations within some finite range. Without this function, the value x_{ij} could diverge. A well-behaved model is also obtained by choosing

$$c(x_{ij}; R) \quad := \quad \begin{cases} 1, & \text{if} \quad -R < x_{ij} < R, \\ 0, & \text{else.} \end{cases} \quad (4)$$

Notice that $N > 2$ since at least three nodes are necessary. The essence of the structure of c is that the relation between the i-th node and the j-th node is influenced by the k-th node, as seen in (1). For example, for $N = 3$ and $c_{ij} = 1$, $i, j = 1, 2, 3$, we have

$$\begin{aligned} \dot{x}_{12} &= x_{13} x_{32}, \\ \dot{x}_{13} &= x_{12} x_{23}, \\ \dot{x}_{23} &= x_{21} x_{13}, \end{aligned}$$

with the symmetry condition $x_{ij} = x_{ji}$. Then in any stationary state the positive product of $x_{ik} x_{kj}$ forces x_{ij} to increase; and similarly with the negative values. This is exactly the condition of the Heider balance; see also [8].

Notice that in a fully connected network with N nodes, the total number of relations N_r and triads of relations $N_\triangle$ are given by

$$N_r := \frac{N(N-1)}{2},$$

and

$$N_\triangle := \frac{N(N-1)(N-2)}{6},$$

respectively.

In the next section, we analyze the stability of the HB system, which is a significant property for studying controllability. An investigation of controllability and of optimal control of a similar problem is proposed in [22]. However, the major difference is that the dynamics of our system has bounded solutions, that is not considered in [22]. Therefore, the spectral analysis proposed in [22] does not apply in our case.

2.1 Stability of the HB model

In this section, we discuss stability of (1). In accordance to the Heider theory, a stable state of the HB model is defined as the balance in the triad of relation $\triangle_{ijk}$ between individuals i, j and k. The balance of a triad is determined by the product of the values of the corresponding edges as follows.

Definition 1. *The triad $\triangle_{ijk}$ of relationship between agents i, j and k is* balanced *if $x_{ij}x_{jk}x_{ki} > 0$, that is,*

$$\mathrm{sign}(x_{ij}x_{jk}x_{ki}) = 1, \tag{5}$$

for any $i, j, k = 1, ..., N$ pairwise distinct, otherwise the triad is unbalanced.

Concerning the evolution of the HB model towards a balanced state, we have the following result.

Proposition 1. *If $x_{ij}(0) \geq -R$, $i, j = 1, \ldots, N$, then there exits $T > 0$ such that all triads become balanced, that is, the product of links on triads $\triangle_{ijk}$ is positive*

$$x_{ij}(T)x_{jk}(T)x_{ki}(T) > 0$$

for $i, j, k = 1, ..., N$ pairwise distinct.

Proof. Consider

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\substack{k=1\\k\neq i,j}}^{N} x_{ij}x_{jk}x_{ki}\\
=\;& \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\substack{k=1\\k\neq i,j}}^{N} \left(\dot{x}_{ij}x_{jk}x_{ki}+x_{ij}\dot{x}_{jk}x_{ki}+x_{ij}x_{jk}\dot{x}_{ki}\right)\\
=\;& \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\dot{x}_{ij}\left(\sum_{\substack{k=1\\k\neq i,j}}^{N} x_{jk}x_{ki}\right)+\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\dot{x}_{jk}\left(\sum_{\substack{i=1\\i\neq j,k}}^{N} x_{ji}x_{ik}\right)\\
&+\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N}\dot{x}_{ki}\left(\sum_{\substack{j=1\\j\neq i,k}}^{N} x_{kj}x_{ji}\right)\\
=\;& \left(\frac{1}{N-2}\right)\left(\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left(1-\frac{x_{ij}^2}{R^2}\right)\left(\sum_{\substack{k=1\\k\neq i,j}}^{N} x_{ik}x_{kj}\right)^2+\sum_{j=1}^{N}\sum_{\substack{k=1\\j\neq j}}^{N}\left(1-\frac{x_{jk}^2}{R^2}\right)\left(\sum_{\substack{i=1\\i\neq j,k}}^{N} x_{ji}x_{ik}\right)^2\right)\\
&+\left(\frac{1}{N-2}\right)\left(\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N}\left(1-\frac{x_{ik}^2}{R^2}\right)\left(\sum_{\substack{j=1\\j\neq i,k}}^{N} x_{ij}x_{jk}\right)^2\right).
\end{aligned}
$$

We have the following cases,

Case I : $|x_{ij}| < R$, for $i, j = 1, ..., N$ and $i \neq j$.

In this case $\frac{d}{dt}(x_{ij}x_{jk}x_{ki}) > 0$, which implies that the product of links in each triads is increasing. Then there exists $T > 0$ such that $x_{ij}(T) = R$ and $\frac{d}{dt}(x_{ij}(t)x_{jk}(t)x_{ki}(t))\Big|_{t=T} = 0$.

Case II : $x_{ij} > R$, for $i, j = 1, ..., N$ and $i \neq j$.

In this case $\frac{d}{dt}(x_{ij}x_{jk}x_{ki}) < 0$, which implies that the product of links in each triads is decreasing. Therefore there exists $T > 0$ such that $\frac{d}{dt}(x_{ij}(t)x_{jk}(t)x_{ki}(t))\Big|_{t=T} = 0$, that is $x_{ij} = R$.

Case III : $x_{ij} < -R$, for $i, j = 1, ..., N$ and $i \neq j$.

In this case $\frac{d}{dt}(x_{ij}x_{jk}x_{ki}) < 0$, which implies that the product of links in each triads is decreasing, then the value of x_{ij} is decreasing and $\lim_{t\to\infty} x_{ij}(t) = -\infty$.

□

The following Proposition establishes the asymptotic behavior of a balanced HB model.

Proposition 2. *If the HB model (1) is balanced, then*

$$\lim_{t\to\infty} x_{ij}(t) = R \quad \text{or} \quad \lim_{t\to\infty} x_{ij}(t) = -R, \tag{6}$$

for $i, j = 1, ..., N$ *and* $i \neq j$.

Proof. Consider

$$V(\mathbf{x}) \;=\; \frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(x_{ij}^2 - R^2)^2.$$

We have

$$\begin{aligned}
\frac{dV(\mathbf{x})}{dt} &= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{1}{2}\frac{d}{dt}(x_{ij}^2 - R^2)^2 \\
&= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(x_{ij}^2 - R^2)\,(2x_{ij}\dot{x}_{ij}) \\
&= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(x_{ij}^2 - R^2)\left(2x_{ij}\left(\frac{1}{N-2}\right)\left(1-\frac{x_{ij}^2}{R^2}\right)\sum_{\substack{k=1\\k\neq i,j}}^{N} x_{ik}x_{kj}\right) \\
&= -\frac{2}{N-2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{(x_{ij}^2 - R^2)^2}{R^2}\sum_{\substack{k=1\\k\neq i,j}}^{N} x_{ij}x_{ik}x_{kj}\right).
\end{aligned}$$

Since (1) is balanced, every triads $\triangle_{ijk}$ is balanced, that is,

$$x_{ij}x_{ik}x_{kj} > 0,$$

for $i, j, k = 1, ..., N$ pairwise distinct. In particular $\dfrac{dV(\mathbf{x})}{dt} < 0$, which concludes the proof. □

As a result of Proposition 1 and Proposition 2, if an edge starts with a value greater than or equal to $-R$, then the HB model reaches a balanced state where the trajectories of relationships may divide into two groups, one of them asymptotically reaches the value R and the other the opposite attains value $-R$. Notice that even if some initial elements x_{ij} are zero, soon they become finite, as long as the initial graph is connected. This is a consequence of the Heider balance rules. Then, the condition of all-to-all coupling is not a limitation. However, it is seen that if the graph is initially divided into separated pieces, then they remain separated.

Next, we study the dynamics of (1) in the neighborhood of the equilibrium points $\mathbf{x}_1^* = \bar{R}$ and $\mathbf{x}_2^* = -\bar{R}$, where $\bar{R} = (R, ..., R) \in \mathbb{R}^{N_r}$. For this purpose, it is convenient to represent the HB model in the following form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= F(\mathbf{x}), \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{7}$$

where $\mathbf{x} = (x_{12}, x_{13}, ..., x_{1N}, x_{23}, ..., x_{2N}, ..., x_{(N-1)N}) \in \mathbb{R}^{N_r}$ and $F(\mathbf{x}) = (f_{12}(\mathbf{x}), ..., f_{1N}(\mathbf{x}), f_{23}(\mathbf{x}), f_{2N}(\mathbf{x}), ..., f_{(N-1)N}(\mathbf{x}))^T$.

The linearized HB model can be written as follows

$$\dot{\mathbf{x}} = A_n \mathbf{x},$$

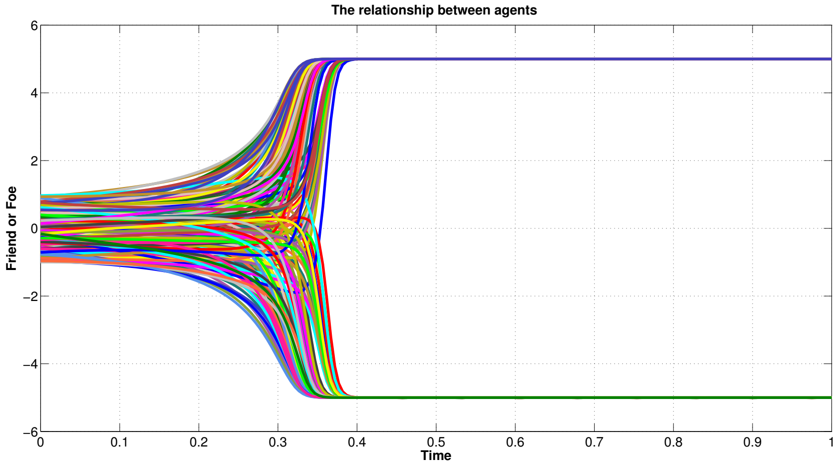
for $n = 1, 2$, where A_1 and A_2 denote the Jacobian matrix of F with respect to $\mathbf{x}$ at $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$, respectively. They are given by

$$\begin{aligned}
\nabla_{\mathbf{x}} F(\mathbf{x}^*) &= \begin{pmatrix} \frac{\partial f_{12}}{\partial x_{12}}(\mathbf{x}^*) & \frac{\partial f_{12}}{\partial x_{13}}(\mathbf{x}^*) & \cdots & \frac{\partial f_{12}}{\partial x_{(N-1)N}}(\mathbf{x}^*) \\ \frac{\partial f_{13}}{\partial x_{12}}(\mathbf{x}^*) & \frac{\partial f_{13}}{\partial x_{13}}(\mathbf{x}^*) & \cdots & \frac{\partial f_{13}}{\partial x_{(N-1)N}}(\mathbf{x}^*) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_{(N-1)N}}{\partial x_{12}}(\mathbf{x}^*) & \frac{\partial f_{(N-1)N}}{\partial x_{13}}(\mathbf{x}^*) & \cdots & \frac{\partial f_{(N-1)N}}{\partial x_{(N-1)N}}(\mathbf{x}^*) \end{pmatrix}, \\
A_1 &= \nabla_{\mathbf{x}} F(\mathbf{x}_1^*) = \begin{pmatrix} -2R & 0 & \cdots & 0 \\ 0 & -2R & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -2R \end{pmatrix}, \\
A_2 &= \nabla_{\mathbf{x}} F(\mathbf{x}_2^*) = \begin{pmatrix} 2R & 0 & \cdots & 0 \\ 0 & 2R & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 2R \end{pmatrix}.
\end{aligned}$$

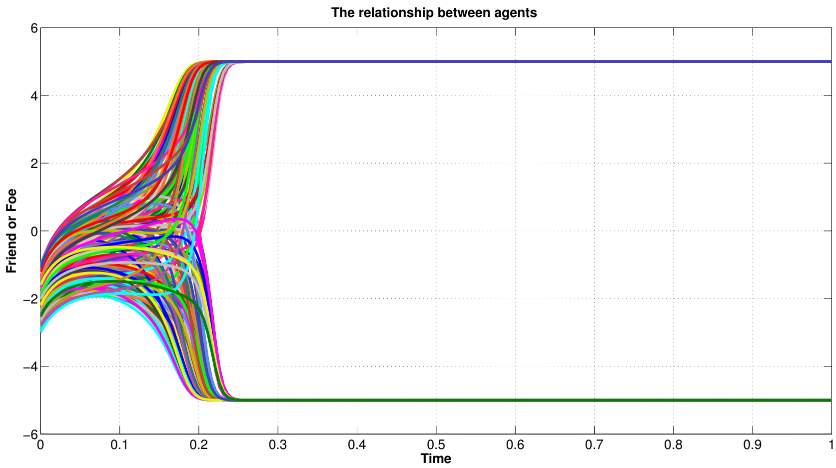
Notice that with A_1, all eigenvalues of the linearized system are strictly less than zero and therefore the equilibrium point $\mathbf{x}_1^* = \bar{R}$ is asymptotically stable while the equilibrium point $\mathbf{x}_2^* = -\bar{R}$ is unstable since all eigenvalues of linearized system about $\mathbf{x}_2^* = -\bar{R}$ are strictly greater than zero.

To give experimental evidence of the theoretical results discussed above, in Figures 1(a) and 1(b) we show numerical results of the HB model with two different initial configurations. We chose $R = 5$. In Figure 1(a), at initial time t_0, values of relationships are distributed between $(-5, 5)$, while in Figure 1(b) all agents in the network start with hostility. Additional details of results of these experiments are given in Table 1 and Table 2, respectively.

We can see from Figures 1(a) and 1(b) that individuals adjust their relationship so that the social group is balanced at final time. Moreover, as predicted by Proposition 2, when the HB model reaches the balance, the final states of relation are divided into two groups, one of them arrives to R, the other meets $-R$. Furthermore, we investigate the behavior of the HB system where some initial relations are zero, that is, at the beginning, we assume that some agents do not know each other or their relationships are unclear. It can be seen in Figure 2 and corresponding Table 3 that as long as the graph is connected, relations between agents are developed and then the system becomes balanced in final time.



(a) $x_{ij}(0) \in (-5, 5)$.



(b) $x_{ij}(0) \in (-5, 0)$.

Figure 1: Simulation with $N = 20$ agents. The status of relation of individuals in Figure (a) is started with friendship or hostility. Figure (b) shows relationships of all agents beginning with hostility.

	initial time	final time
total triads($N_\triangle$)	1140	1140
balanced triads($N_{\triangle_b}$)	532	1140
unbalanced triads ($N_{\triangle_{ub}}$)	608	0

Table 1: The number of triads of relationship between agents corresponding to Figure 1(a).

	initial time	final time
total triads($N_\triangle$)	1140	1140
balanced triads($N_{\triangle_b}$)	0	1140
unbalanced triads ($N_{\triangle_{ub}}$)	1140	0

Table 2: The number of triads of relationship between agents corresponding to Figure 1(b).

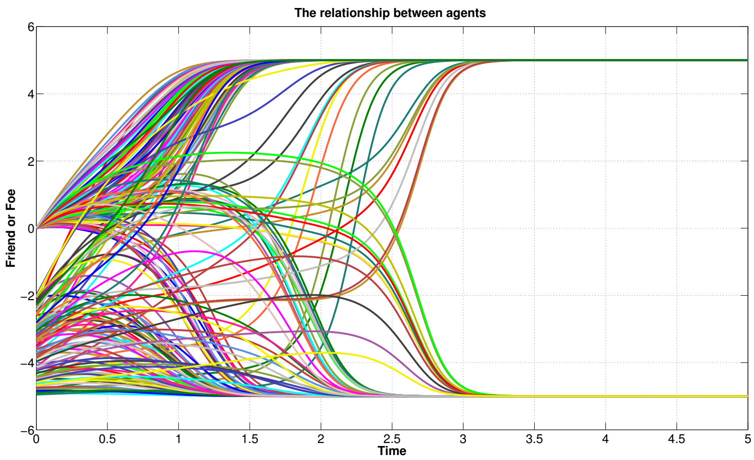
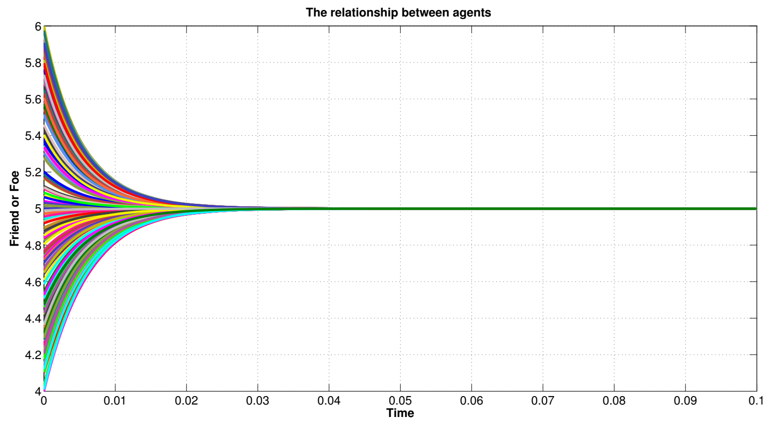


Figure 2: Simulation with $N = 20$ agents. Initially, relationships of individuals are divided into two groups, one of hostility and the other of different values.

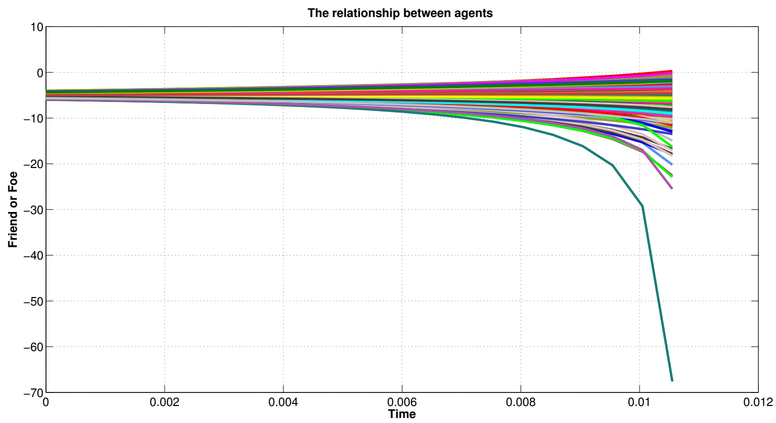
	initial time	final time
total triads($N_\triangle$)	1140	1140
balanced triads($N_{\triangle_b}$)	0	1140
unbalanced triads ($N_{\triangle_{ub}}$)	285	0
triads containing unclear-relationship links ($N_{\triangle_{nor}}$)	855	0

Table 3: The number of triads of relationship between agents corresponding to Figure 2.

Next, we investigate numerically the stability properties of the HB model. Figure 3(a) shows that $\mathbf{x}_1^* = R$ is asymptotically stable since trajectories starting in a neighborhood of R asymptotically reach this point. Conversely, in Figure 3(b), taking a starting value close to the equilibrium point $\mathbf{x}_2^* = -R$, we obtain trajectories diverging from $-R$.
We remark that the dynamics of the HB model is confined in $-R \le x_{ij} \le R$. As proved in Proposition 1, the HB model always tends to balance, that is a stable configuration of the system. In addition, Proposition 1 also investigate stability in the case $x_{ij}(0) > R$ and $x_{ij}(0) < -R$, respectively. In the former case, $x_1^* = \bar{R}$ results asymptotically stable; see also Figure 3(a). In the latter case, $x_2^* = -\bar{R}$ is unstable for links starting with $x_{ij}(0) < -R$; see Figure 3(b).



(a) $x_{ij}(0) \in [4, 6]$.



(b) $x_{ij}(0) \in [-6, -4]$.

Figure 3: Simulation with $N = 20$ agents. Figure(a) shows the solution of the HB model where the relations of all agents begin with friendship. Figure (b) shows the relationship of all agents starting with hostility.

3 Local Controllability of the HB model

Consider the control of the HB model where the controlling agent is linked to all agents of the network. The resulting system of relationship of N interacting agents together with one reference agent is governed by the following set of differential equations

$$\begin{aligned} \dot{x}_{0i}(t) &= u_i(t), \\ \dot{x}_{ij}(t) &= \frac{1}{N-2}\left(1-\frac{x_{ij}^2(t)}{R^2}\right)\sum_{\substack{k=1\\k\neq i,j}}^{N} x_{ik}(t)x_{kj}(t)+\gamma x_{0i}(t)x_{0j}(t), \end{aligned} \tag{8}$$

for $i,j=1,...,N$, with given initial relationships $x_{ij}(t_0)=x_{ij}^0$. The index 0 denotes the leader, and the index i denotes the i-th individual in the network. The variables x_{0i}, $i=1,...,N$, denote the relationships between the leader and the other individuals, while x_{ij} represent relationships between individuals in the community. The function $u_i(t)\in L^2((0,T),\mathbb{R})$ is the control and the parameter $\gamma>0$ is added in order to avoid divergence of states. We notice that the model (8) now has $N_r=\frac{(N+1)N}{2}$ equations, whereas $N_c=N$ equations are related to the controlling links. We denote $N_{uc}=N_r-N_c$ the number of uncontrolled links.

In this section, we discuss the local controllability for the HB system. Consider the linearization of system (8) around the equilibrium points $\mathbf{x}_1^*=R$ and $\mathbf{x}_2^*=-R$. The linearized system for the variable $\tilde{\mathbf{x}}=\mathbf{x}-\mathbf{x}^*$ is given by

$$\dot{\tilde{\mathbf{x}}}=A\tilde{\mathbf{x}}+Bu, \tag{9}$$

where A is a block matrix and B is a block vector as follows

$$A=\begin{pmatrix}0_{N_c,N_c} & 0_{N_c,N_{uc}}\\ L & D\end{pmatrix},\qquad B=\begin{pmatrix}I_{N_c,N_c}\\ 0_{N_{uc},N_c}\end{pmatrix}, \tag{10}$$

where $0_{n,m}$ is a $n\times m$ null matrix and $I_{n,n}$ denote an $n\times n$ identity matrix, $L\in\mathbb{R}^{N_{uc}\times N_c}$ and $D\in\mathbb{R}^{N_{uc}\times N_{uc}}$ are presented as follows

$$\begin{aligned} L &= \gamma\begin{pmatrix} x^* & x^* & 0 & \cdots & 0\\ x^* & 0 & x^* & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ x^* & 0 & 0 & \cdots & x^*\\ 0 & x^* & x^* & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & x^* & x^* \end{pmatrix}_{N_{uc},N_c}, \\ D &= \begin{pmatrix} -2x^* & 0 & \cdots & 0\\ 0 & -2x^* & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & -2x^* \end{pmatrix}_{N_{uc},N_{uc}}. \end{aligned}$$

We can see that rank of the Kalmann matrix $K(A,B)$

$$\begin{aligned} K(A,B) &= \begin{bmatrix} B & AB & A^2B & \dots & A^{Nr-1}B \end{bmatrix} \\ &= \left[\begin{pmatrix} I_{N_c,N_c} \\ 0_{N_{uc},N_c} \end{pmatrix} \begin{pmatrix} 0_{N_c,N_c} \\ L_{N_{uc},N_c} \end{pmatrix} \begin{pmatrix} 0_{N_c,N_c} \\ (DL)_{N_{uc},N_c} \end{pmatrix} \dots \begin{pmatrix} 0_{N_c,N_c} \\ (D^{N_r-2}L)_{N_{uc},N_c} \end{pmatrix} \right] \end{aligned}$$

is equal to $2N_c$, that is, it has full rank if and only if $N_{uc} = N_c$. Therefore in the case of three agents and one leader (with $N_r = 6$, $N_c = 3$, and $N_{uc} = 3$) is system (8) is locally controllable around the equilibria R and $-R$. Indeed A and B are given by

$$A = \begin{pmatrix} 0_{3,3} & 0_{3,3} \\ L & D \end{pmatrix}, \qquad B = \begin{pmatrix} I_{3,3} \\ 0_{3,3} \end{pmatrix},$$

with the matrix

$$L = \gamma \begin{pmatrix} x^* & x^* & 0 \\ x^* & 0 & x^* \\ 0 & x^* & x^* \end{pmatrix}, \qquad D = \begin{pmatrix} -2x^* & 0 & 0 \\ 0 & -2x^* & 0 \\ 0 & 0 & -2x^* \end{pmatrix}.$$

Therefore the Kalmann matrix given by

$$\begin{aligned} K(A,B) &= \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \\ &= \left[\begin{pmatrix} I_{3,3} \\ 0_{3,3} \end{pmatrix} \begin{pmatrix} 0_{3,3} \\ L \end{pmatrix} \begin{pmatrix} 0_{3,3}, \\ DL \end{pmatrix} \begin{pmatrix} 0_{3,3} \\ D^2L \end{pmatrix} \begin{pmatrix} 0_{3,3} \\ D^3L \end{pmatrix} \begin{pmatrix} 0_{3,3} \\ D^4L \end{pmatrix} \right]. \end{aligned}$$

has full rank. In general, in all the other configurations we cannot infer any controllability property from the analysis of the linearized system.

4 A HB optimal control problem

In this section, we formulate an optimal control problem of the HB model with the presence of a leader, that is
Minimize

$$J(x,u) = \frac{1}{2}\sum_{i=1}^{N}(x_{ij}(T) - x_{des}(T))^2 + \frac{\nu}{2}\int_0^T \|\mathbf{u}(t)\|^2 dt, \tag{11}$$

subject to the differential constraint given by

$$\begin{aligned} \dot{x}_{0i}(t) &= u_i(t), \\ \dot{x}_{ij}(t) &= \frac{1}{N-2}\left(1 - \frac{x_{ij}^2}{R^2}\right)\sum_{k=0}^{N} x_{ik}x_{kj} + \gamma x_{i0}x_{j0}, \\ x_{ij}(0) &= x_{ij}^0. \end{aligned} \tag{12}$$

for $i,j = 1,...,N$. This optimal control problem requires to find a vector of controls $u_i : (0,T) \to \mathbb{R}$, $i = 1,\dots,N$, such that the HB model evolves from the given initial condition to a final state $x_{0i}(T)$ that is as close as possible

to the desired state $x_{des}(T)$ while minimizing the cost of the control given by the second term of the cost functional J, where $\nu > 0$ represents the weight of the cost of the control. We denote $\|\mathbf{u}(t)\|^2 = u_1^2(t) + \ldots + u_N^2(t)$.
In order to solve (11) - (12), we discretize (12) using the Runge-Kutta (RK) discretization scheme proposed in [11, 12]. This choice guarantees a high-order approximation of the HB model that is suitable to construct a discretization scheme for the adjoint HB problem that an exact numerical gradient is obtained. See [5, 24] for successful applications of this scheme to discretize flocking and opinion forming optimal control problems.
In the following, we illustrate this particular RK scheme to approximate the optimality system that characterizes the optimal solution to our optimal control problem. For this purpose, we reformulate (11)-(12) as the following general optimal control problem

$$\begin{aligned} \min \quad J(\mathbf{x}, \mathbf{u}) &= \phi(\mathbf{x}(T)) \\ \text{subject to} \quad \dot{\mathbf{x}}(t) &= F(\mathbf{x}(t), \mathbf{u}(t)), \quad t \in [0, T] \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{13}$$

where $\mathbf{x}(t) \in H^1((0,T);\mathbb{R}^{N_r})$ and $\mathbf{u}(t) \in \mathbb{R}^{N_c}$ are called the state and control variables, respectively. We choose $\mathbf{u} \in L^2((0,T);\mathbb{R}^{N_c})$. The function $\phi : \mathbb{R}^{N_r} \to \mathbb{R}$ represents the objective and the dynamic of the model is given by $F : \mathbb{R}^{N_r} \times \mathbb{R}^{N_c} \to \mathbb{R}^{N_r}$. We assume that for a given $\mathbf{u}$ the dynamical model in (13) admits a unique solution $\mathbf{x} = \mathbf{x}(\mathbf{u})$ and the map $\mathbf{u} \mapsto \mathbf{x}(\mathbf{u})$ is differentiable.
We introduce the following equation

$$\begin{aligned} \dot{\hat{x}}(t) &= \frac{\nu}{2}\|\mathbf{u}(t)\|^2. \\ \hat{x}(t_0) &= 0, \end{aligned}$$

so that the optimal control problem (11)-(12) can be written as

$$\begin{aligned} \min \quad J(\tilde{\mathbf{x}}, \mathbf{u}) &= \frac{1}{2}\sum_{i=1}^{N}(x_{ij}(T) - x_{des}(T))^2 + \hat{x}(T) \\ \text{subject to} \quad \dot{x}_{0i} &= u_i(t) \\ \dot{x}_{ij} &= \frac{1}{N-2}\left(1 - \frac{x_{ij}^2}{R^2}\right)\sum_{k=0}^{N} x_{ik}x_{kj} + \gamma x_{0i}x_{0j}, \quad \text{for} \quad i = 1, ..., N, \\ \dot{\hat{x}} &= \frac{\nu}{2}\|\mathbf{u}(t)\|^2. \end{aligned} \tag{14}$$

with given initial conditions. For the ease of notation, we can write (14) as follows

$$\begin{aligned} \min \quad J(\tilde{\mathbf{x}}, \mathbf{u}) &= \frac{1}{2}\sum_{i=1}^{N}(x_{ij}(T) - x_{des}(T))^2 + \hat{x}(T) \\ \text{subject to} \quad \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \mathbf{u}), \end{aligned} \tag{15}$$

where $\tilde{\mathbf{x}} = (x_{01}, x_{02}, ..., x_{N(N-1)}, \hat{x})^T \in \mathbb{R}^{N_r+1}$. We consider the discretization of the optimality system (15) by a RK scheme on a uniform time mesh, with the following time-step size

$$h = \frac{T}{n},$$

where n is the total number of discrete time intervals in $(0, T)$ and the value of $\tilde{\mathbf{x}}(t)$ at the discrete time t_k is denoted with

$$\tilde{\mathbf{x}}_k = \tilde{\mathbf{x}}(t_k), \quad t_k = kh, \quad \text{for} \quad k = 0, \dots, n.$$

Corresponding to the RK discretization setting, the optimal control problem (15) with s-stage RK scheme becomes the following

$$\begin{aligned} \min \quad J(\tilde{\mathbf{x}}, \mathbf{u}) &= \frac{1}{2}\sum_{i=1}^{N}(x_{ij}(T) - x_{des}(T))^2 + \hat{x}(T) \\ \text{subject to} \quad \tilde{\mathbf{x}}_{k+1} &= \tilde{\mathbf{x}}_k + h\sum_{i=1}^{s} b_i\tilde{\mathbf{f}}(\mathbf{y}_i, \mathbf{u}_{ki}), \quad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0, \\ \mathbf{y}_{ki} &= \tilde{\mathbf{x}}_k + h\sum_{j=1}^{s} a_{ij}\tilde{\mathbf{f}}(\mathbf{y}_{,kj}, \mathbf{u}_{kj}), \end{aligned} \tag{16}$$

for $\tilde{n} = 1, 2$, $1 \leq i, j \leq s$, and $0 \leq k \leq n - 1$.
where the vector $\mathbf{u}_k \in \mathbb{R}^{N_c \times s}$ represents the s-stages of the RK discrete control vector at time step k. We have

$$\mathbf{u}_k = (\mathbf{u}_{k1}, \mathbf{u}_{k2}, ..., \mathbf{u}_{ks}) \in \mathbb{R}^{N_c \times s}.$$

Summing-up, the RK discretization of the optimal control problem of the HB system are governed by the following

$$\begin{aligned} \min \quad J(\tilde{\mathbf{x}}, \mathbf{u}) &= \frac{1}{2}\sum_{i=1}^{N}(x_{ij}(T) - x_{des}(T))^2 + \hat{x}(T) \\ \text{subject to} \quad \tilde{\mathbf{x}}_{k+1} &= \tilde{\mathbf{x}}_k + h\sum_{i=1}^{s} b_i\tilde{\mathbf{f}}(\mathbf{y}_i, \mathbf{u}_{ki}), \quad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0, \\ \mathbf{y}_{ki} &= \tilde{\mathbf{x}}_k + h\sum_{j=1}^{s} a_{ij}\tilde{\mathbf{f}}(\mathbf{y}_{,kj}, \mathbf{u}_{kj}), \end{aligned} \tag{17}$$

The discrete optimality system corresponding to (16) is given by

$$\begin{aligned} \tilde{\mathbf{x}}_{k+1} &= \tilde{\mathbf{x}}_k + h\textstyle\sum_{i=1}^{s} b_i\tilde{\mathbf{f}}(\mathbf{y}_{ki}, \mathbf{u}_{ki}), \qquad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0, \\ \mathbf{y}_{ki} &= \tilde{\mathbf{x}}_k + h\textstyle\sum_{j=1}^{n} a_{ij}\tilde{\mathbf{f}}(\mathbf{y}_{kj}, \mathbf{u}_{kj}), \\ \Psi_k &= \Psi_{k+1} + \textstyle\sum_{i=1}^{s} b_i\chi_{ki}, \qquad \Psi_n = -\nabla_x\phi(\tilde{\mathbf{x}}_n), \\ \chi_{ki} &= (\nabla_x\tilde{\mathbf{f}}(\mathbf{y}_{ki}, \mathbf{u}_{ki}))^\top \left(\Psi_{k+1} + \textstyle\sum_{j=1}^{s} \frac{b_j a_{ij}}{b_i}\chi_{kj}\right). \end{aligned} \tag{18}$$

From this system, the following gradient results

$$\nabla_{\mathbf{u}_{ki}} J(u) = -(\nabla_u \tilde{\mathbf{f}}(\mathbf{y}_{ki}, \mathbf{u}_{ki}))^\top \left(\Psi_{k+1} + \sum_{j=1}^{s} \frac{b_j a_{ij}}{b_i} \chi_{kj} \right). \tag{19}$$

for $1 \leq i, j \leq s$, and $0 \leq k \leq n-1$.

To investigate the well-posedness of the optimal control problem (16), we remark that the smoothness and coercivity conditions stated in [11] can be verified for the optimal control problems governed by the HB system (16). It follows that the accuracy results stated in Theorem 2.1 in [11] for the RK scheme applied to (16) hold.

Next, we discuss a model predictive control (MPC) scheme (see, for instance Ref. [9]) implementing a closed-loop control strategy for the HK model in order to track a given sequence of desired configurations in time. Let $(0, T)$ be the time interval where the evolution is considered. We assume time windows of size $\Delta t = T/M$ for positive integer M. Let $t_m = m\Delta t$, $m = 0, 1, \ldots, M$. At time t_0, we have given initial conditions denoted with $\tilde{\mathbf{x}}_0$. We also have the desired positions at the end of each time window $x_{des}(t_m)$, $m = 1, \ldots, M$. Our MPC strategy starts at time t_0 and solves the open-loop optimal control problem (16) defined in the interval (t_0, t_1). Then, the results $\tilde{\mathbf{x}}$ of system measured in time $t = t_1$ will be a initial value for the subsequent optimization problem defined in the interval (t_1, t_2). This procedure is repeated by receding the time horizon until the last time window is reached. We notice that the closed-loop system with the MPC scheme is nominally asymptotically stable; see [9].
The MPC procedure is summarized in the following algorithm.

Algorithm 1 (MPC Control)**.** *Set $m = 0$, $\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0$;*

1. *measure the state $\tilde{\mathbf{x}}(t_m) = \tilde{\mathbf{x}}_m$ and the target $x_{des}(t_{m+1})$;*
2. *in (t_m, t_{m+1}), set initial condition $\tilde{\mathbf{x}}_m^0 = \tilde{\mathbf{x}}_m$;*
3. *solve (16), thus obtain the optimal pair $(\tilde{\mathbf{x}}, u)$;*
4. *If $t_{m+1} < T$, set $m := m + 1$, $\tilde{\mathbf{x}}_m = \tilde{\mathbf{x}}(t_m)$, go to 1.*
5. *End.*

Concerning the third step of Algorithm 1, consisting in solving the optimal control problem (16), notice that the solution of the state equation in (16) gives the mapping $\mathbf{u} \to \tilde{\mathbf{x}}(\mathbf{u})$, that allows to transform the constrained optimization problem in an unconstrained one as follows

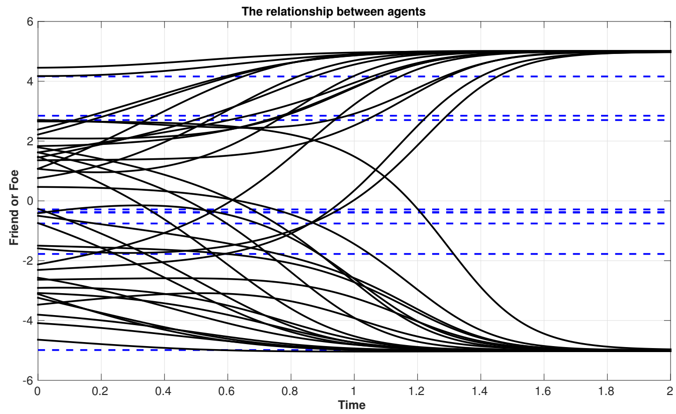
$$\min_{\mathbf{u} \in U} J(\mathbf{u}) := J(\tilde{\mathbf{x}}(\mathbf{u}), \mathbf{u}). \tag{20}$$

We solve these problems implementing a nonlinear conjugate gradient strategy. The evaluation of the corresponding gradient is given in (19): For a given u, we solve first the forward equation and then the adjoint problem. We solve (16) implementing the gradient in a nonlinear conjugate gradient (NCG) scheme; see, e.g., Ref. [4]. For details on NCG implementation see, for instance, Refs. [11, 12].

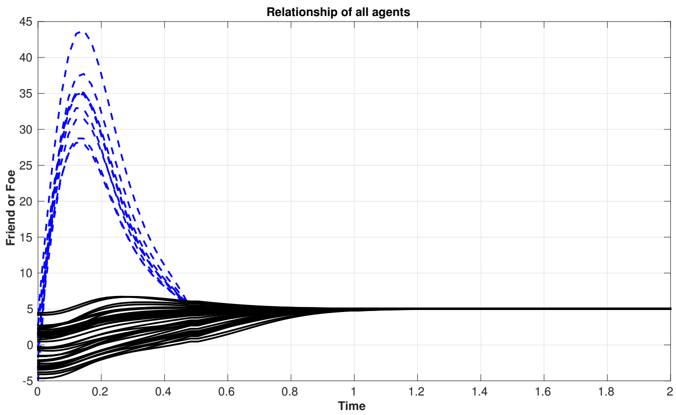
5 Numerical experiments

The objective of this section is to present results of numerical experiments with our HB optimal control problem. We chose the time horizon $T = 2$. The objective is to find the optimal control in order to drive the HB system to reach a friendship state where $x_{ij} = R$ for all $i, j = 0, ..., N,\ i \neq j$. We consider three series of experiments; in the first one, the initial conditions $x_{ij}(0) \in (-5, 5)$. In the second one, the state of relations starts with hostility, that is, in a neighborhood of the unstable equilibrium point $\mathbf{x}^* = -\bar{R}$. In the third one, at the initial time, relationships between leader and agents in network are given to be zeros, otherwise they are randomly chosen between friendly or hostile. For all cases, we solve the optimal control problem (11) with $N = 9$ individuals and one leader, with $R = 5$. In the objective functional we take $\nu = 0.001$. Furthermore, the target is $x_{des} = R$. To apply the MPC strategy, the time horizon is divided into subintervals of size $\Delta t = 0.25$.

Case I The initial state of relationships of agents in our network is randomly chosen with friendly and hostile values. With this set of initial conditions, we get the results shown in Figure 4, where Figure 4(a) shows the solution of the HB model with zero control $u_{0i} = 0$, for $i = 1, ..., N$, and in Figure 4(b) the controllers are activated into the HB system. As we see in Figure 4(a), the HB model evolves towards the equilibrium states as expected because the value of the controls u_{0i} is equal to zero. Therefore the leader has no influence on the other agents and fails in steering the agents to a friendship state. On the other hand, as we see in Figure 4(b), as soon as the control is active, friendship is obtained.



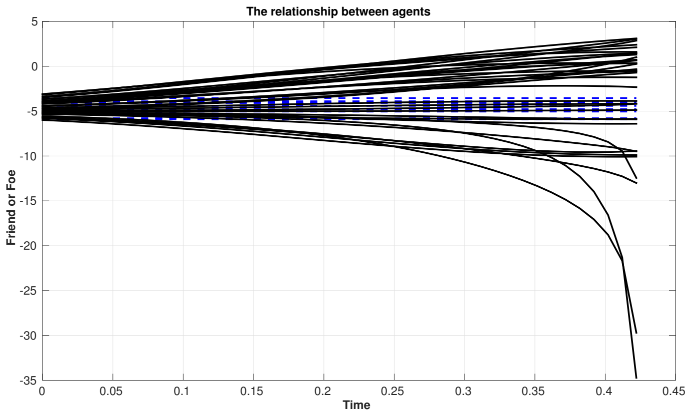
(a) $x_{ij}(0) \in [-5, 5]$ with no controller.



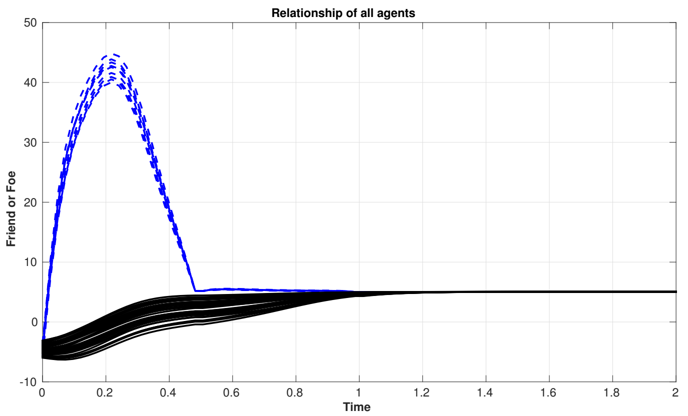
(b) $x_{ij}(0) \in [-5, 5]$ with controllers.

Figure 4: Simulation with $N + 1 = 10$ agents. The status of relation of individuals in Figure (a) are started randomly with friendship or hostility, $x_{ij} \in (-5, 5)$. Figure (a) shows the results where no controller is included in the system while in Figure (b) controllers are included in the HB system. The dot-lines represent state of the relationship of leader and normal individuals, otherwise are of normal individuals.

Case II In this case, the initial state of relationships of the agents in our network is randomly placed in a neighborhood of the unstable hostile equilibrium point, $\mathbf{x}* = -R$. With this set of initial conditions, we get the results as shown in Figure 5. Specifically in Figure 5(a), we see that the solution of the HB model without active control is unstable. On the other hand, whenever the leader is actively controlling the system, the HB system successfully reaches the desired friendship state; see Figure 5(b).



(a) $x_{ij}(0) \in [-6, 3]$ with no controller.

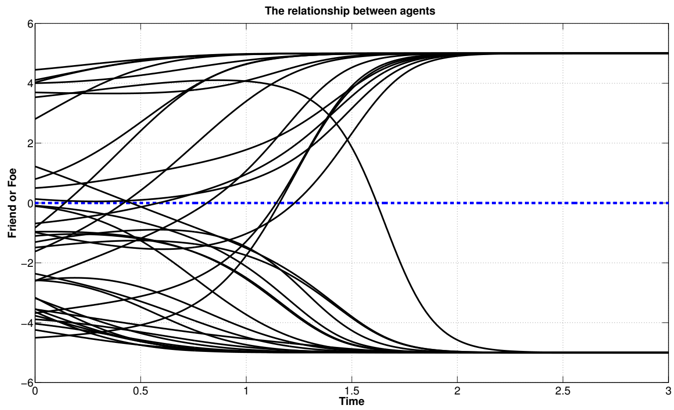


(b) $x_{ij}(0) \in [-6, -3]$ with controllers.

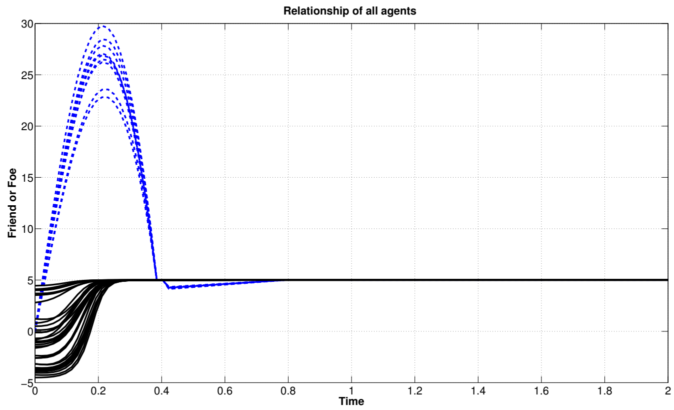
Figure 5: Simulation with $N + 1 = 10$ agents. The status of relation of individuals in Figure (a) is started with hostility $x_{ij} \in (-6, 3)$. Figure (a) shows the results where no controller is included the system while in Figure (b) controllers are included in the HB system. The dot-lines represent state of the relationship of leader and normal individuals, otherwise are of normal individuals.

Case III In this case, at the initial time the status of relationships between leader and agents in network is zero, $x_{oi}^0 = 0$, while the state of relationships between other agents in the network are randomly chosen friendship, hostility, or unclear relationship. Figure 6(a) shows the results of the HB system when controls are not included in the system. However, when the controls enter in the system, the status of relationships of all agents in network are forced to reach the friendship as seen in Figure 6(b).

We remark that results of numerical experiments show that our control strategy is ineffective in driving the HB model to hostility.



(a) $x_{0i}(0) = 0$ and $x_{ij} \in [-5, 5]$ with no controller.



(b) $x_{0i}(0) = 0$ and $x_{ij} \in [-5, 5]$ with controllers.

Figure 6: Simulation with $N + 1 = 10$ agents. The status of relation of leader and individuals in Figure (a) is started with $x_{0i} = 0$, for $i = 1, ..., N$, whereas another status is randomly given between friendship, hostility or no relation. Figure (a) shows the results where no controller is included the system while in Figure (b) controllers are included in the HB system. The dot-lines represent state of the relationship of leader and normal individuals, otherwise are of normal individuals.

6 Conclusion

The continuous time Heider balance model and related control issues are investigated. The HB model describes the evolution of relationship in a social network. It is shown that in the absence of controls, this model evolves towards equilibrium states of friendship and/or hostility. In correspondence to these states the local stability of the linearized system is discussed. Furthermore, an optimal control strategy steering the relationships in the network to a desired friendship state is investigated. The corresponding optimization problems are solved with an appropriate Runge–Kutta method that guarantees accurate gradients of the objectives. These gradients are imple-

mented in a nonlinear conjugate gradient solution procedure and a model predictive control scheme. Results of numerical experiments demonstrate the effectiveness of the proposed control strategy.
It is proved that the HB model evolves to reach a balanced configuration. We expect this behavior to persist also if small perturbations in the model are introduced. Nevertheless, it would be interesting to study a stochastic extension of the HB model in the spirit of [6, 10, 21].

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