Sparse Control of Alignement models

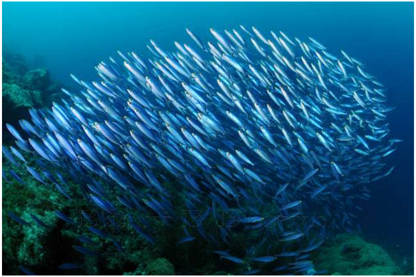
Marco Caponigro

Équipe M2N
Conservatoire National des Arts et Métiers
Paris

Mathematical Control in Trieste
SISSA, December $2^{nd} - 6^{th}$, 2013

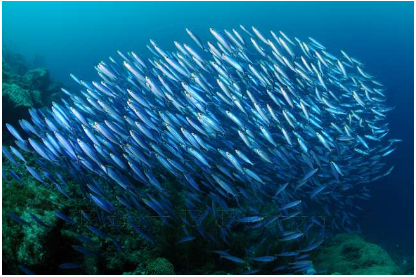
Emergent behaviors in multi-agent systems

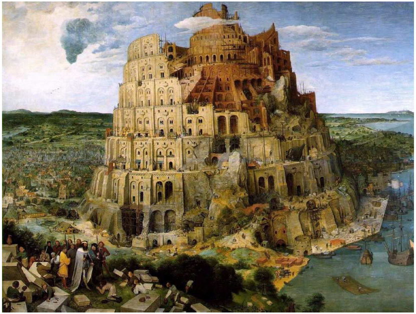
- Birds Flocking; Fish Schooling; Locusts Swarming;



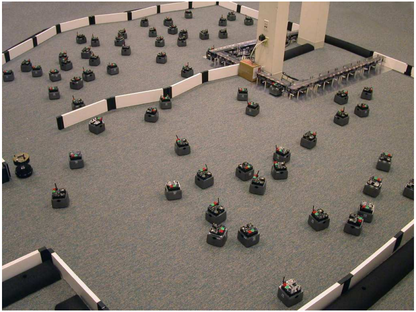
Emergent behaviors in multi-agent systems

- Birds Flocking; Fish Schooling; Locusts Swarming;
- Social dynamics; Language Evolution.





Emergent behaviors in multi-agent systems



- Birds Flocking; Fish Shooling; Locusts Swarming;
- Social dynamics; Language Evolution.

Applications:

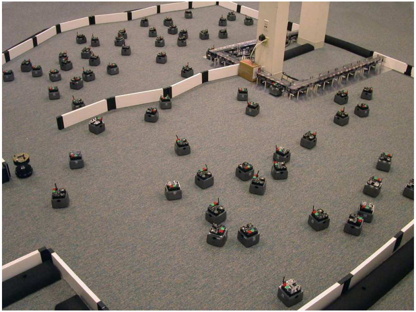
- Forcing emergent behaviors in swarms of robots;

Emergent behaviors in multi-agent systems

- Birds Flocking; Fish Shooling; Locusts Swarming;
- Social dynamics; Language Evolution.

Applications:

- Forcing emergent behaviors in swarms of robots;
- Avoid financial crisis or “Black Swans”.



The Cucker–Smale model

Cucker–Smale (2007)

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i) \end{cases} \qquad (CS)$$

where $x_1, \ldots, x_N, v_1, \ldots, v_N \in \mathbb{R}^d$ and a is positive and nonincreasing.

- $x_i \in \mathbb{R}^d$ is the *main state* (e.g. position);
- $v_i \in \mathbb{R}^d$ is the *consensus parameter* (e.g. velocity);
- $a(\cdot)$ represents *the communication rate*.

  In Cucker–Smale (2007) : $a(x) = \dfrac{1}{(1+x^2)^\beta}, \; \beta > 0.$

The Cucker–Smale model

Cucker–Smale (2007)

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i) \end{cases} \tag{CS}$$

where $x_1, \ldots, x_N, v_1, \ldots, v_N \in \mathbb{R}^d$ and a is positive and nonincreasing.

In matrix notation:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -L_x v, \end{cases}$$

where L_x is the Laplacian. [1]

[1] Given a nonnegative symmetric $N \times N$ matrix $A = (a_{ij})_{i,j}$, the *Laplacian* L of A is defined by $L = D - A$, with $D = \operatorname{diag}(d_1, \ldots, d_N)$ and $d_k = \sum_{j=1}^{N} a_{kj}$.

Consensus

- The mean velocity $\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i$ is conserved.

Definition (Consensus point)

A steady configuration of System (CS) $(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ where

$$v_1 = \ldots = v_N$$

is called a *consensus point*.

- The dynamics originating from a consensus point (x, v) is given by rigid translation $x(t) = x + t\bar{v}$.

Unconditional consensus emergence

- the bilinear form $B(u,v) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle$,
- the *dispersion* $X(t) := B(x(t), x(t)) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2$,
- the *disagreement* $V(t) := B(v(t), v(t)) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2$.

Unconditional consensus emergence

- the bilinear form $B(u,v) = \frac{1}{2N^2}\sum_{i,j=1}^{N}\langle u_i - u_j, v_i - v_j\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle u_i, v_i\rangle - \langle \bar{u}, \bar{v}\rangle$,
- the *dispersion* $X(t) := B(x(t), x(t)) = \frac{1}{2N^2}\sum_{i,j=1}^{N}\|x_i(t) - x_j(t)\|^2$,
- the *disagreement* $V(t) := B(v(t), v(t)) = \frac{1}{2N^2}\sum_{i,j=1}^{N}\|v_i(t) - v_j(t)\|^2$.

Theorem (Cucker–Smale '07; Ha–Tadmor '08; Carrillo et al. '08)

If

$$a(x) = \frac{1}{(1+x^2)^\beta} \quad \text{and} \quad \beta \leq \frac{1}{2},$$

every solution of (CS) tend to consensus.

- Moreover if $\beta < 1/2$ there exists $\alpha > 0$ such that for every $t > 0$,

$$X(t) \leq \bar{X} \quad \text{and} \quad V(t) \leq V(0)e^{-t\alpha}$$

Conditional consensus

Proposition (Ha–Ha–Kim, 2010)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *be such that* $X_0 = B(x_0, x_0)$ *and* $V_0 = B(v_0, v_0)$ *satisfy*

$$\sqrt{V_0} \leq \int_{\sqrt{X_0}}^{\infty} a(\sqrt{2N}r)dr\,.$$

Then the solution of (CS) with initial data (x_0, v_0) *tends to consensus.*

Non-Consensus

Two agents moving on $\mathbb{R}$ with $a = 2/(1+x^2)$.

- $x(t) = x_1(t) - x_2(t)$ relative main state
- $v(t) = v_1(t) - v_2(t)$ relative consensus parameter

Then

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\dfrac{v}{1+x^2} \end{cases}$$

with $x(0) = x_0$ and $v(0) = v_0 > 0$. The solution of this system is

$$v(t) - v_0 = -\arctan x(t) + \arctan x_0.$$

If $\arctan x_0 + v_0 > \pi/2$ ($\Longrightarrow$ $\arctan x_0 + v_0 \geq \pi/2 + \varepsilon$ for some $\varepsilon > 0$) then

$$v(t) \geq -\arctan x(t) + \pi/2 + \varepsilon > \varepsilon,$$

for every $t > 0$. In other words, the system does **not** tend to consensus.

Self-organization

Admissible controls, accounting for the external field, are measurable functions $u = (u_1, \ldots, u_N) : [0, +\infty) \to (\mathbb{R}^d)^N$ satisfying the $\ell_1^N - \ell_2^d$-norm constraint

$$\sum_{i=1}^{N} \|u_i(t)\| \leq M,$$

for every $t > 0$, for a given $M > 0$.

Cucker–Smale System

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) \end{cases} \qquad (CCS)$$

for $i = 1, \ldots, N$, and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$.

Self-organization Vs Organization via intervention

Admissible controls, accounting for the external field, are measurable functions $u = (u_1, \ldots, u_N) : [0, +\infty) \to (\mathbb{R}^d)^N$ satisfying the $\ell_1^N - \ell_2^d$-norm constraint

$$\sum_{i=1}^{N} \|u_i(t)\| \leq M,$$

for every $t > 0$, for a given $M > 0$.

Controlled Cucker–Smale System

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t), \end{cases} \qquad (CCS)$$

for $i = 1, \ldots, N$, and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$.

Totally distributed control

Proposition

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exists* $T > 0$ *and* $u : [0, T] \to (\mathbb{R}^d)^N$ *with* $\sum_{i=1}^N \|u_i(t)\| \leq M$ *for every* $t \in [0, T]$ *such that the associated solution tends to consensus*

Totally distributed control

Proposition

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exists* $T > 0$ *and* $u : [0, T] \to (\mathbb{R}^d)^N$ *with* $\sum_{i=1}^N \|u_i(t)\| \leq M$ *for every* $t \in [0, T]$ *such that the associated solution tends to consensus*

Consider the solution with initial data (x_0, v_0) associated with the feedback

$$u(t) = -\alpha(v(t) - \bar{v}(t)) \qquad \text{with } 0 < \alpha \leq \frac{M}{N\sqrt{B(v_0, v_0)}}.$$

Then,

$$\begin{aligned}
\frac{d}{dt}V(t) &= \frac{d}{dt}B(v(t), v(t)) \\
&= -2B(L_x v(t), v(t)) + 2B(u(t), v(t)) \\
&\leq 2B(u(t), v(t)) = -2\alpha B(v - \bar{v}, v - \bar{v}) = -2\alpha V(t).
\end{aligned}$$

Therefore $V(t) \leq e^{-2\alpha t}V(0)$ and $V(t) \to 0$ exponentially fast as $t \to \infty$.

Greedy control: the variational principle

For every $(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and $M > 0$, let $U(x, v)$ be defined as the set of solutions of the variational problem

$$\min \quad B(u, v)$$

$$\text{subject to } \sum_{i=1}^{N} \|u_i\| \leq M \,,$$

Greedy control: the variational principle

For every $(x,v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and $M > 0$, let $U(x,v)$ be defined as the set of solutions of the variational problem

$$\min \left(B(u,v) + \frac{\gamma(x)}{N} \sum_{i=1}^{N} \|u_i\| \right)$$

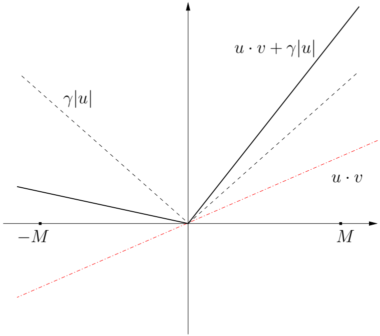
$$\text{subject to } \sum_{i=1}^{N} \|u_i\| \leq M \,,$$

where

$$\gamma(x) = \int_{\sqrt{B(x,x)}}^{\infty} a(\sqrt{2N}r)dr.$$

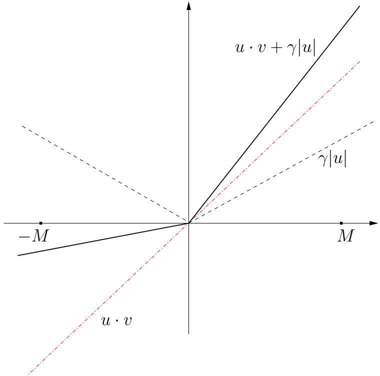
Geometric interpretation (scalar case)

Case $|v| < \gamma$



$u = 0$ unique solution in $[-M, M]$

Case $|v| > \gamma$



$|u| = M$ unique solution in $[-M, M]$

Greedy controls are stabilizating

Theorem

For every initial pair $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$*, the differential inclusion*

$$(\dot{x}, \dot{v}) \in \{(v, -L_x v + u) \mid u \in U(x, v)\}$$

with initial condition $(x(0), v(0)) = (x_0, v_0)$ *is well-posed and its solutions converge to consensus as* t *tends to* $+\infty$*.*

Greedy controls are stabilizating

Theorem

For every initial pair $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$*, the differential inclusion*

$$(\dot{x}, \dot{v}) \in \{(v, -L_x v + u) \mid u \in U(x, v)\}$$

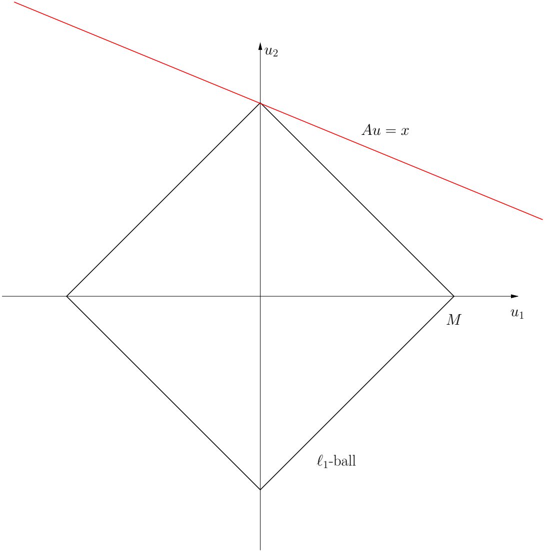
with initial condition $(x(0), v(0)) = (x_0, v_0)$ *is well-posed and its solutions converge to consensus as* t *tends to* $+\infty$*.*

For almost every $(x, v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ the solution of

$$\min \left(B(u, v) + \frac{\gamma(x)}{N} \sum_{i=1}^{N} \|u_i\| \right) \text{ subject to } \sum_{i=1}^{N} \|u_i\| \leq M \,,$$

is **sparse**, in particular it has at most **one nonzero component**.

Sparsity: the geometric interpretation



Consider the case $d = 1$, $N = 2$. We want to find

$$\max\{Au = x\},$$

subject to

$$|u_1| + |u_2| \leq M.$$

Sparse control: explicit construction

Let $v_\perp(t) = v(t) - \bar{v}(t)$.

- If $\|v_{\perp_i}\| \leq \gamma(x)$ for every $i = 1, \ldots, N \implies$ consensus region reached $\implies u_1 = \cdots = u_N = 0$.

Sparse control: explicit construction

Let $v_\perp(t) = v(t) - \bar{v}(t)$.

The sparse control

- If $\|v_{\perp_i}\| \leq \gamma(x)$ for every $i = 1, \ldots, N \implies$ consensus region reached $\implies u_1 = \cdots = u_N = 0$.
- Otherwise let $j \in \{1, \ldots, N\}$ be the smallest index such that

$$\|v_{\perp_j}\| = \max_{1 \leq i \leq N} \|v_{\perp_i}\|$$

then

$$u_j = -M \frac{v_{\perp_j}}{\|v_{\perp_j}\|}, \quad \text{and} \quad u_i = 0 \quad \text{for every } i \neq j.$$

Sparse control: explicit construction

Let $v_\perp(t) = v(t) - \bar{v}(t)$.

The sparse control or "Shepherd dog" strategy

- If $\|v_{\perp_i}\| \leq \gamma(x)$ for every $i = 1, \ldots, N \implies$ consensus region reached $\implies u_1 = \cdots = u_N = 0$.
- Otherwise let $j \in \{1, \ldots, N\}$ be the smallest index such that

$$\|v_{\perp_j}\| = \max_{1 \leq i \leq N} \|v_{\perp_i}\|$$

then

$$u_j = -M \frac{v_{\perp_j}}{\|v_{\perp_j}\|}, \quad \text{and} \quad u_i = 0 \quad \text{for every } i \neq j.$$

(Loading 4 agents)

(Loading 5 agents)

(Loading 6 agents)

(Loading 20 agents)

(Loading 50 agents)

Optimality of the greedy strategy

All the controls in $U(x,v)$ are of the form

$$u_i = \begin{cases} 0 & \text{if } v_{\perp_i} = 0 \\ -\alpha_i \frac{v_{\perp_i}}{\|v_{\perp_i}\|} & \text{if } v_{\perp_i} \neq 0 \end{cases}$$

for $\alpha_i \geq 0$, $\sum_{i=1}^N \alpha_i \leq M$.

Proposition

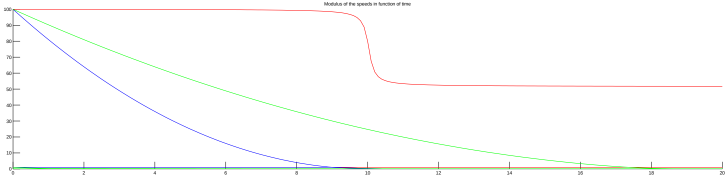
The sparse feedback control u*, associated with the solution* $((x(t), v(t))$ *is a minimizer of*

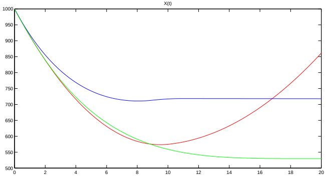
$$\mathcal{R}(t,u) = \frac{d}{dt} V(t),$$

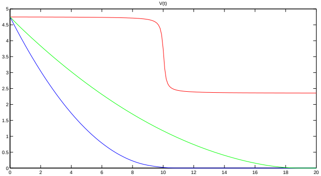
over all possible feedback controls in $U(x(t), v(t))$*. In other words, the sparse feedback control* u *is the best choice in terms of the rate of convergence to consensus.*

Numerical simulations I

- $N = 20$;
- $x_1(0) = (-100, 0)$ and $v_1(0) = (-1, 0)$;
- $x_2(0) = (100, 0)$ and $v_2(0) = (-10, 0)$;
- $x_3(0) = \ldots = x_{20}(0) = (0, 0) = v_3(0) = \cdots = v_{20}(0)$;

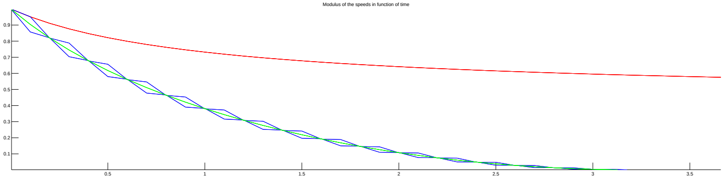


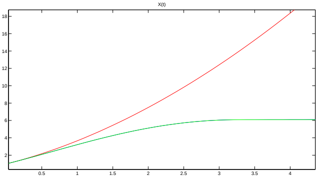


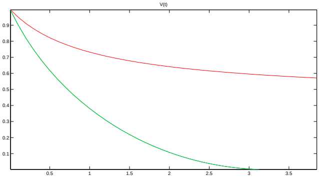


Numerical simulations II: Fully symmetric case

- $N = 4$
- $x_1(0) = (-1, 0)$ and $v_1(0) = (-1, 0)$;
- $x_2(0) = (0, 1)$ and $v_2(0) = (0, 1)$
- $x_3(0) = (1, 0)$ and $v_3(0) = (1, 0)$
- $x_4(0) = (0, -1)$ and $v_4(0) = (0, -1)$







Sample and hold

Definition (Sampling solution)

Let $U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ be continuous and locally Lipschitz in x uniformly on compact subset of $\mathbb{R}^n \times U$. Given a feedback $u : \mathbb{R}^n \to U$, $\tau > 0$, and $x_0 \in \mathbb{R}^n$ we define the *sampling solution* of the differential system

$$\dot{x} = f(x, u(x)), \quad x(0) = x_0,$$

as the continuous (actually piecewise C^1) function $x : [0, T] \to \mathbb{R}^n$ solving recursively for $k \geq 0$

$$\dot{x}(t) = f(x(t), u(x(k\tau))), \quad t \in [k\tau, (k+1)\tau]$$

using as initial value $x(k\tau)$, the endpoint of the solution on the preceding interval, and starting with $x(0) = x_0$. We call τ the *sampling time*.

Time sparse control

Theorem

Fix $M > 0$ and consider the sparse feedback control u. Then for every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ there exists $\tau_0 > 0$ small enough, such that for all $\tau \in (0, \tau_0]$ the sampling solution of (CCS) associated with the control u, the sampling time τ, and initial pair (x_0, v_0) reaches the consensus region in finite time.

In particular the systems reaches the consensus region within time

$$T_0 = \frac{2N}{M}(\sqrt{V(0)} - \gamma(\bar{X}))$$

where $\bar{X} = 2B(x_0, x_0) + \frac{2N^4}{M^2}B(v_0, v_0)^2$

Local Controllability

Let $\mathcal{V}_f = \{(v_1, \dots, v_N) \in (\mathbb{R}^d)^N \mid v_1 = \cdots = v_N\}$.

Theorem

- *For every* $(\tilde{x}_0, \tilde{v}_0) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$*,*
- *for almost every* $(\tilde{x}_1, \tilde{v}_1) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$*,*
- *for every* $\delta > 0$*, and*
- *for every* $i = 1, \dots, N$

there exist $T > 0$ *and a control* $u : [0, T] \to [0, \delta]^d$ *steering the system from* $(\bar{x}, \bar{v})$ *to* $(\tilde{x}, \tilde{v})$*, with the property* $u_j(t) = 0$ *for every* $j \neq i$ *and every* $t \in [0, T]$*.*

Local Controllability

Let $\mathcal{V}_f = \{(v_1, \dots, v_N) \in (\mathbb{R}^d)^N \mid v_1 = \cdots = v_N\}$.

Theorem

- *For every* $(\tilde{x}_0, \tilde{v}_0) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$,
- *for almost every* $(\tilde{x}_1, \tilde{v}_1) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$,
- *for every* $\delta > 0$, *and*
- *for every* $i = 1, \dots, N$

there exist $T > 0$ *and a control* $u : [0, T] \to [0, \delta]^d$ *steering the system from* $(\bar{x}, \bar{v})$ *to* $(\tilde{x}, \tilde{v})$, *with the property* $u_j(t) = 0$ *for every* $j \neq i$ *and every* $t \in [0, T]$.

$$\begin{cases} \dot{x}^k = v^k \\ \dot{v}^k = -L_{\bar{x}} v^k + Bu \,, \end{cases}$$

for every $k = 1, \dots, d$ where $B = (1, 0, \dots, 0)^t$
Therefore we reduce the investigation of the Kalman condition for a linear system on $\mathbb{R}^N$ of the form

$$\dot{v} = (-L_{\bar{x}})v + Bu.$$

Global sparse controllability

Corollary

For every $M > 0$*, for every initial condition* $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$*, for almost every* $(x_1, v_1) \in (\mathbb{R}^d)^N \times \mathcal{V}_f$*, there exist* $T > 0$ *and a componentwise and time sparse control* $u : [0, T] \to (\mathbb{R}^d)^N$*, satisfying* $\sum_{i=1}^N \|u_i(t)\| \leq M$*, such that the corresponding solution starting at* (x_0, v_0) *arrives at the consensus point* (x_1, v_1) *within time* T*.*

Optimal Control

We consider the optimal control problem

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t), \end{cases} \tag{CCS}$$

with running cost, for $\gamma > 0$,

$$\int_0^T \left(\sum_{i=1}^{N} \left(v_i(t) - \frac{1}{N} \sum_{j=1}^{N} v_j(t) \right)^2 + \gamma \sum_{i=1}^{N} \|u_i(t)\| \right) dt.$$

Optimal Control

We consider the optimal control problem

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t), \end{cases} \tag{CCS}$$

with running cost, for $\gamma > 0$,

$$\int_0^T \left(\sum_{i=1}^{N} \left(v_i(t) - \frac{1}{N}\sum_{j=1}^{N} v_j(t) \right)^2 + \gamma \sum_{i=1}^{N} \|u_i(t)\| \right) dt.$$

- High dimensional of the state space $\implies$ difficulties in applying theoretical results (PMP);
- Codimension of the non-sparse manifold in the space of (x, v, p_x, p_v);

(Loading 6 agents - optimal control)

(Loading 6 agents - optimal control)

Reference

- Marco Caponigro (CNAM), Massimo Fornasier (TU Münich), Benedetto Piccoli (Rutgers), Emmanuel Trélat (Paris 6),
  Sparse Stabilization and Control of the Cucker-Smale Model,
  arXiv:1210.5739.

Reference

- Marco Caponigro (CNAM), Massimo Fornasier (TU Münich), Benedetto Piccoli (Rutgers), Emmanuel Trélat (Paris 6),
  Sparse Stabilization and Control of the Cucker-Smale Model,
  arXiv:1210.5739.

Thank you!

Reference

- Marco Caponigro (CNAM), Massimo Fornasier (TU Münich), Benedetto Piccoli (Rutgers), Emmanuel Trélat (Paris 6),
  Sparse Stabilization and Control of the Cucker-Smale Model,
  arXiv:1210.5739.

Thank you!

Conference on **Control of PDEs**

March 31 - April 4, 2014

`http://controlpde2014.cnam.fr`

CNAM - Paris