Controllability on the Group of Diffeomorphisms

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Ph.D thesis
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References

1. Controllability on the Group of Diffeomorphisms,*
   *To appear on Annales IHP.*
   Prep. SISSA 79/2008/M.

2. Dynamics Control by a Time–varying Feedback,*
   *To appear on JDCS.*
   Prep. SISSA 81/2008/M.

3. Families of vector fields which generate the group of
diffeomorphisms,

*= with Andrei A. Agrachev
Let $M$ be a smooth connected manifold. Let $V, V_t$ be complete.

<table>
<thead>
<tr>
<th>Autonomous v.f. $V \in \text{Vec}M$</th>
<th>Nonautonomous v.f. $V_t \in \text{Vec}M$</th>
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</table>
| $\begin{cases} 
\dot{q}(t) &= V(q(t)) \\
q(0) &= q_0 
\end{cases}$ | $\begin{cases} 
\dot{q}(t) &= V_t(q(t)) \\
q(t_0) &= q_0 
\end{cases}$ |

For every fixed $t$

$$e^{tV}$$

\[ \exp \int_{t_0}^{t} V_\tau \, d\tau \]

is a diffeomorphism of $M$ which maps any $q_0 \in M$ to the value of the solution at time $t$ of the system.
Our goal

Let $\mathcal{F} \subseteq \text{Vec}M$ be a family of vector fields we set

$$\text{Gr}\mathcal{F} = \{e^{tf_1} \circ \cdots \circ e^{tf_k} : t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\}.$$ 

Our purpose is to study the relation between $\text{Gr}\mathcal{F}$ and $\text{Diff}_0(M)$.

Thurston 1971

If $M$ is compact then the group $\text{Diff}_0(M)$ is simple.

If $\mathcal{F} = \text{Vec}M$ then $\text{Gr}\mathcal{F}$ is a normal subgroup of $\text{Diff}_0(M)$. Therefore

$$\text{Gr}(\text{Vec}M) = \text{Diff}_0(M)$$
Our goal

Let $\mathcal{F} \subset \text{Vec}M$ be a family of vector fields we set

$$\text{Gr}\mathcal{F} = \{ e^{t_1f_1} \circ \cdots \circ e^{t_kf_k} : t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N} \}.$$  

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**Thurston 1971**

If $M$ is compact then the group $\text{Diff}_0(M)$ is simple.

If $\mathcal{F} = \text{Vec}M$ then $\text{Gr}\mathcal{F}$ is a *normal* subgroup of $\text{Diff}_0(M)$. Therefore

$$\text{Gr} (\text{Vec}M) = \text{Diff}_0(M)$$
The main result

**Theorem**

*If $M$ is compact and $\text{Gr}\mathcal{F}$ acts transitively on $M$, then*

$$\text{Gr}\{af : a \in C^\infty (M), f \in \mathcal{F}\} = \text{Diff}_0 M.$$ 

**Remark (Lobry)**

The set of pairs $(f_1, f_2)$ such that $\text{Gr}\{f_1, f_2\}$ acts transitively on $M$ is dense in $\text{Vec}M \times \text{Vec}M$. 
The main result

Main Theorem

Let $M$ be a compact connected manifold and $\mathcal{F} \subset \text{Vec}M$. If $\text{Gr}\mathcal{F}$ acts transitively on $M$, then there exist

- a neighborhood $\mathcal{O}$ of the identity in $\text{Diff}_0(M)$;
- a positive integer $\mu$

such that every $P \in \mathcal{O}$ can be presented in the form

$$P = e^{a_1 f_1} \circ \ldots \circ e^{a_\mu f_\mu},$$

for some $f_1, \ldots, f_\mu \in \mathcal{F}$ and $a_1, \ldots, a_\mu \in C^\infty(M)$.

Remark

The number of exponentials $\mu$ does not depend on $P$.

Open Problem

Estimate $\mu$. 
The main result

Main Theorem

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Open Problem

Estimate $\mu$. 
By *Control System* we mean a system of the form:

\[ \dot{q} = f_u(q), \quad q \in M, u \in U, \]

where

- \( q \in M \) is called *state*;
- \( u \in U \) is called *control*;
- \( U \subset \mathbb{R}^m \) is called *set of control parameters*.

We represent the control system by a family of vector fields

\[ \mathcal{F} = \{ f_u : u \in U \} \subset \text{Vec}M. \]
The set of points reachable is called *attainable set*.

### Attainable set

\[
A_q = \{ q \circ e^{tf_1} \circ \cdots \circ e^{tf_k} : t_i \geq 0, f_i \in \mathcal{F}, k \in \mathbb{N} \}.
\]

We consider a larger set: the *Orbit*

### Orbit

\[
O_q = \{ q \circ e^{tf_1} \circ \cdots \circ e^{tf_k} : t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N} \} = \{ q \circ P : P \in \text{Gr}\mathcal{F} \}.
\]

If a family \( \mathcal{F} \) is symmetric, namely if \( \mathcal{F} = -\mathcal{F} \), then the attainable sets coincide with the orbits, i.e. \( A_q = O_q \).
Controllability

**Definition: Controllability**

A system $\mathcal{F}$ is *controllable* $\iff A_q = M$, for every $q \in M$.

**Remark**

Gr$\mathcal{F}$ acts transitively on $M$ $\iff O_q = M$, for every $q \in M$.

If $\mathcal{F}$ is symmetric then

Controllability on $M$

$\Downarrow$

“Controllability” on $\text{Diff}_0(M)$
Bracket Generating families

Definition

- $\text{Lie}(\mathcal{F}) = \text{span}\{[f_1, \ldots [f_{k-1}, f_k] \ldots] : f_1, \ldots, f_k \in \mathcal{F}, k \in \mathbb{N}\}$
- $\text{Lie}_q \mathcal{F} = \{f(q) : f \in \text{Lie}(\mathcal{F})\}$.

Definition

We say that the family $\mathcal{F}$ is \textit{bracket generating} if

$$\text{Lie}_q \mathcal{F} = T_q M \quad \text{for every } q \in M.$$ 

Theorem (Chow–Rashevsky)

Let $\mathcal{F}$ be a \textit{bracket generating} family of vector fields. Then

$$O_q = M, \quad \text{for any } q \in M.$$
Application to control systems

Corollary

Let \( \{ f_1, \ldots, f_m \} \) be bracket generating. Consider the system

\[
\dot{q} = \sum_{i=1}^{m} u_i(t, q)f_i, \quad q \in M,
\]  

with controls that are

- piecewise constant in \( t \),
- smooth in \( q \).

For every \( P \in \text{Diff}_0(M) \) there exist controls \( u_i(t, q) \) such that

\[
P = \exp \int_0^1 \sum_{i=1}^{m} u_i(t, \cdot)f_i \, dt.
\]
Outline of the proof

- Localization of the problem;
- Use the controllability assumption to consider a full-dimensional case;
- Restriction to a 1-dimensional problem with parameters;
- Linearize the diffeomorphism.
Localization

**Lemma (Palis–Smale)**

Let $\bigcup_j U_j = M$ be an open covering of $M$ and $\mathcal{O}$ be a neighborhood of identity in $\text{Diff}_0 M$.

Then the group $\text{Diff}_0 M$ is generated by the subset

$$\{ P \in \mathcal{O} : \exists j \text{ such that } \text{supp } P \subset U_j \}.$$

Where $\text{supp } P = \overline{\{ x \in M : P(x) \neq x \}}$. 
Orbit Theorem

**Theorem (Orbit Theorem of Sussmann)**

\[ \mathcal{O}_q \text{ is a connected submanifold of } M. \text{ Moreover,} \]

\[ T_p \mathcal{O}_q = \text{span}\{ q \circ \text{Ad}Pf : P \in \text{Gr}\mathcal{F}, f \in \mathcal{F} \}, \quad p \in \mathcal{O}_q. \]

Recall that transitivity of the action of \( \text{Gr}\mathcal{F} \) on \( M \) \( \implies \mathcal{O}_q = M \).

If \( X_1(q), \ldots, X_n(q) \) form a basis of \( T_qM \) then

\[ e^{a_1X_1} \circ \cdots \circ e^{a_nX_n} \in \text{Gr} \{ af : a \in C^\infty(M), f \in \mathcal{F} \} \]

for every \( a_1, \ldots, a_n \in C^\infty(M) \).

Indeed \( X_i = \text{Ad}P_i f_i \) for \( i = 1, \ldots, n \) with \( P_i \in \text{Gr}\mathcal{F}, f_i \in \mathcal{F} \).
The problem reduces to

Given \( X_1, \ldots, X_n \) such that

\[
\text{span}\{X_1(0), \ldots, X_n(0)\} = \mathbb{R}^n.
\]

We have to prove that there exist

- an open neighborhood \( U \subset \mathbb{R}^n \);
- a open subset of \( \mathcal{O} \subset \text{Diff}_0(U) \);

such that every \( P \in \mathcal{O} \) can be written as

\[
P = e^{a_1X_1} \circ \cdots \circ e^{a_nX_n}.
\] (2)

In the following we study analytical properties of map

\[
(a_1, \ldots, a_n) \mapsto e^{a_1X_1} \circ \cdots \circ e^{a_nX_n} \big|_U.
\]
Outline of the proof

- Localization of the problem;
- Use the controllability assumption to consider a full-dimensional case;
- Restriction to a 1-dimensional problem with parameters;
- Linearize the diffeomorphism.
Restriction to a single direction

Let $X_1, \ldots, X_n \in \text{Vec}\mathbb{R}^n$ such that

$$\text{span}\{X_1(0), \ldots, X_n(0)\} = \mathbb{R}^n.$$ 

For

- $U$ neighborhood of the origin in $\mathbb{R}^n$;
- $\mathcal{U}$ neighborhood of the identity in $\text{Diff}_0(U)$;

small enough every $P \in \mathcal{U}$ splits into the composition

$$P = \varphi_1 \circ \cdots \circ \varphi_n \big|_U,$$

where $\varphi_i \in \text{Diff}(U)$ and preserves the 1-foliation generated by the trajectories of the equation $\dot{q} = X_i(q)$, for every $i = 1, \ldots, n$. Namely of the form

$$\varphi_i = \exp \int_0^1 a(t, \cdot)X_i\, dt.$$
Idea

The linear diffeomorphism of $U \subset \mathbb{R}$, say

$$x \mapsto \alpha x\big|_U, \quad \alpha \neq 1, \quad (\alpha > 0),$$

is the exponential of the linear vector field $\log(\alpha)x\frac{\partial}{\partial x}$.

- It is possible to take a nonempty open subset of $U$ such that the linearization of every $\varphi_k$ is not trivial.
- The change of coordinates that linearizes can be recovered from the solution of the PDE:

$$a(t, x, y)\frac{\partial u}{\partial x}(t, x, y) + \frac{\partial u}{\partial t}(t, x, y) + b(t, x, y)u(t, x, y) = 0, \quad (3)$$

with $t, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}$.
Improvements

\[ \dot{q} = \sum_{i=1}^{m} u_i(t, q)f_i, \quad q \in M, \]

with \( \{f_1, \ldots, f_m\} \) bracket generating.

For every \( P \in \text{Diff}_0(M) \) there exist controls \( u_i(t, q) \) that are

(i) piecewise constant w.r.t. \( t \),

(ii) smooth w.r.t. \( q \).

such that \( P \) is the flow at time 1 of the system.

- Is it possible to assume controls more regular?
- Is it possible to add a drift to the system?
The Second Result

Let \( \{f_1, f_2, \ldots, f_m\} \) bracket generating. Consider the system

\[
\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t, q)f_i(q), \quad q \in \mathbb{R}^n,
\]

with controls \( u_i \) such that, for every \( i = 1, \ldots, m \):

(i) \( u_i \) is polynomial with respect to \( q \in \mathbb{R}^n \);
(ii) \( u_i \) is a trigonometric polynomial with respect to \( t \in [0, 1] \).

Let \( k \) be a positive integer and consider

\[
J_k^0(P)(z) = P(0) + (DP(0)) \cdot z + \frac{1}{2} (D^2P(0)) \cdot z^{\otimes 2} + \cdots + \frac{D^kP(0)}{k!} \cdot z^{\otimes k}.
\]

Let \( r \) be a positive integer, \( \varepsilon > 0 \), and \( B \) ball in \( \mathbb{R}^n \). For any \( P \in \text{Diff}_0(\mathbb{R}^n) \), there exist controls \( u_1(t, q), \ldots, u_m(t, q) \) such that, if \( \Phi \) is the flow at time 1 of the system then

\[
J_k^0(\Phi) = J_k^0(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.
\]
The Second Result

Let \( \{f_1, f_2, \ldots, f_m\} \) bracket generating. Consider the system

\[
\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t, q)f_i(q), \quad q \in \mathbb{R}^n,
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J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.
\]
The core of the method

We have to study analytical properties of map

\[(a_1, \ldots, a_n) \mapsto e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n} \big|_U.\]  

(4)

Consider

- the space \(X\) of polynomials of degree \(\leq k\), in \(n\) variables;
- the jet–group \(Y = J^k_0(\text{Diff}_0(\mathbb{R}^n))\);

and consider the map:

\[
F : \quad X^n \quad \mapsto \quad Y
\]

\[
(a_1, \ldots, a_n) \quad \mapsto \quad J^k_0(e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n})
\]

\[\dim X < \infty \text{ and } \dim Y < \infty\]
Implicit Function Theorem applied to $F$

\[ F : \mathbb{X}^n \rightarrow \mathbb{Y} \]
\[ (a_1, \ldots, a_n) \mapsto J^k_0(e^{a_1X_1} \circ \cdots \circ e^{a_nX_n}) \]

- $F(0, \ldots, 0) = \text{Id}$;
- $T_{\text{Id}}\mathbb{Y} = J^k_0(\text{Vec}(\mathbb{R}^n))$;
- $D_0F(a_1, \ldots, a_n) = a_1J^k_0(X_1) + \cdots + a_nJ^k_0(X_n)$.

$D_0F$ is surjective

Thus $F$ is locally surjective.
Moreover:
- $F$ is continuous;
- $F$ has a right inverse;
- the right inverse of $F$ is continuous.
**Relaxation**

**Theorem**

If $\text{Gr}\mathcal{F}$ acts transitively on $\mathbb{R}^n$. For any $P \in \text{Diff}_0(\mathbb{R}^n)$, there exists a sequence

$$\{P_j\}_j \subset \text{Gr}\{af : a \in C^\infty(\mathbb{R}^n), f \in \mathcal{F}\}$$

such that

$$P_j \to P, \quad \text{as } j \to \infty$$

in the $C^\infty$–topology.

**Proposition**

If $V_t = \sum_{i=1}^{m} a_i(t, \cdot)X_i$, $\Rightarrow \exists Z^n_t$ sequence of piecewise constant w.r.t. $t$ vector fields s.t. $Z^n_t \in \{aX_i | a \in C^\infty, i = 1 \ldots, m\}, \quad \forall t, n,$ and

$$\exp \int_0^t Z^n_{\tau} \, d\tau \longrightarrow \exp \int_0^t V_{\tau} \, d\tau, \quad \text{as } n \to \infty$$

in the $C^\infty$–topology and uniformly w.r.t. $t \in [0, 1]$. 
Relaxation

Theorem

If $\text{Gr}\mathcal{F}$ acts transitively on $\mathbb{R}^n$. For any $P \in \text{Diff}_0(\mathbb{R}^n)$, there exists a sequence

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$$\overrightarrow{\exp} \int_0^t Z^n_{\tau} \, d\tau \longrightarrow \overrightarrow{\exp} \int_0^t V_{\tau} \, d\tau, \quad \text{as } n \to \infty$$

in the $C^\infty$–topology and uniformly w.r.t. $t \in [0, 1]$. 
\[a_1(t, \cdot)X_1\]  
\[\exp \int_0^t V_\tau \, d\tau\]

\[a_2(t, \cdot)X_2\]

\[\exp \int_0^t Z_\tau \, d\tau\]

\[\exp \int_0^t Z_{\tau+1} \, d\tau\]
Let \( \{f_1, \ldots, f_m\} \) be a bracket–generating family and consider the control–affine system

\[
\dot{q} = \sum_{i=1}^{m} u_i(t, q)f_i(q), \quad q \in \mathbb{R}^n.
\]

for every \( P \in \text{Diff}_0(\mathbb{R}^d) \):

- there exist \( u_i(t, \cdot) \) piecewise constant in \( t \)

such that

\[
J^k_0(P) = J^k_0(e^{a_1X_1} \circ \ldots \circ e^{a_nX_n}) = J^k_0 \left( \lim_{\varepsilon \to 0} \exp \int_0^1 \sum_{i=1}^{m} u_i(t, \cdot)f_i \, dt \right).
\]

and

\[
\lim_{\varepsilon \to 0} \exp \int_0^1 \sum_{i=1}^{m} u_i(t, \cdot)f_i \, dt \text{ is arbitrary close to } P.
\]
Lemma

Consider the control system

\[ \dot{q} = \sum_{i=1}^{m} u_i(t, q)f_i(q), \quad q \in \mathbb{R}^n, \]

with

- \( \{f_1, f_2, \ldots, f_m\} \) bracket generating;
- \( u_i \) piecewise constant with respect to \( t \in [0, 1] \);
- \( u_i \) smooth with respect to \( q \).

Let \( N \) and \( r \) be positive integers, \( \varepsilon > 0 \), and \( B \) ball in \( \mathbb{R}^n \). For any \( P \in \text{Diff}_0(\mathbb{R}^n) \), there exist controls \( u_1(t, q), \ldots, u_m(t, q) \) such that, if

\[ \Phi = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t, \cdot)f_i \, dt. \]

then

\[ J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon. \]
If $U$ is the space of controls $u(t, q)$:
- smooth w.r.t. $q$;
- piecewise constant w.r.t. $t$.

By Implicit Function Theorem the map:

$$\tilde{F} : \quad U^m \quad \longrightarrow \quad \mathbf{Y}$$

$$(u_1, \ldots, u_m) \quad \longmapsto \quad J^k_0(\exp \int_0^1 \sum_{i=1}^m u_i(t, \cdot) f_i \, dt)$$

is continuous, surjective and with continuous right inverse.

**Remark**

Let $\varepsilon > 0$. If $G : U^m \rightarrow \mathbf{Y}$ is s.t. $\sup_{x \in K} |\tilde{F}(x) - G(x)| < \varepsilon$ for any $K$ compact, then $G$ is surjective too.

Small perturbations of map $\tilde{F}$ remain surjective.
Theorem

Let \( \{f_1, f_2, \ldots, f_m\} \) be a bracket generating family of vector fields on \( \mathbb{R}^n \). Consider the control system

\[
\dot{q} = \sum_{i=1}^{m} u_i(t, q) f_i(q), \quad q \in \mathbb{R}^n,
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with controls \( u_i \) such that, for every \( i = 1, \ldots, m \):

(i) \( u_i \) is smooth w.r.t. \( q \in \mathbb{R}^n \);

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Let \( N \) and \( r \) be positive integers, \( \varepsilon > 0 \), and \( B \) ball in \( \mathbb{R}^n \). For any \( P \in \text{Diff}_0(\mathbb{R}^n) \), there exist controls \( u_1(t, q), \ldots, u_m(t, q) \) such that, if

\[
\Phi = \exp \int_0^1 f_0 + \sum_{i=1}^{m} u_i(t, \cdot) f_i \, dt.
\]

then

\[
J^k_0(\Phi) = J^k_0(P) \quad \text{and} \quad \| \Phi - P \|_{C^r(B)} < \varepsilon.
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Theorem

Let \( \{f_1, f_2, \ldots, f_m\} \) be a bracket generating family of vector fields on \( \mathbb{R}^n \). Consider the control system

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Let \( \{f_1, f_2, \ldots, f_m\} \) be a bracket generating family of vector fields on \( \mathbb{R}^n \). Consider the control system

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(i) \( u_i \) is polynomial w.r.t. \( q \in \mathbb{R}^n \);

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Let \( N \) and \( r \) be positive integers, \( \varepsilon > 0 \), and \( B \) ball in \( \mathbb{R}^n \). For any \( P \in \text{Diff}_0(\mathbb{R}^n) \), there exist controls \( u_1(t, q), \ldots, u_m(t, q) \) such that, if

\[
\Phi = \exp \int_0^1 f_0 + \sum_{i=1}^{m} u_i(t, \cdot) f_i \, dt.
\]

then

\[
J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.
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Theorem

Let \( \{f_1, f_2, \ldots, f_m\} \) be a bracket generating family of vector fields on \( \mathbb{R}^n \). Consider the control system

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\]

then

\[
J^k_0(\Phi) = J^k_0(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.
\]
An application of Nash–Moser

\[ F : \ C^\infty(M)^n \rightarrow \text{Diff}_0(M) \]

\[ (a_1, \ldots, a_n) \rightarrow e^{a_1X_1} \circ \cdots \circ e^{a_nX_n} \]  \hspace{1cm} (5)

The problem is to prove:

1. \( F \) is locally onto;
2. small perturbations of the map \( F \) are locally surjective too.

Remark

Recall that point 1 implies the Main Theorem.
An alternative proof of Main Theorem

Proposition

Let \( X_i \in \text{Vec}\mathbb{R}^n \), \( i = 1, \ldots, n \), such that

\[
\text{span}\{X_1(0), \ldots, X_n(0)\} = \mathbb{R}^n.
\]

Then, there exist \( \varrho > 0 \) and an open subset \( \mathcal{U} \subset C^\infty_0(B_{\varrho})^n \), such that the mapping

\[
F : \mathcal{U} \rightarrow \text{Diff}_0(B_{\varrho}), \quad (a_1, \ldots, a_n) \mapsto (e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n})\big|_{B_{\varrho}},
\]

is an open map from \( \mathcal{U} \) into \( \text{Diff}_0(B_{\varrho}) \), where

\[
B_{\varrho} = \{ e^{s_1 X_1} \circ \cdots \circ e^{s_n X_n}(0) : |s_i| < \varrho, \ i = 1, \ldots, d \}.
\]
It is possible to prove that

1. $F$ maps $C^k$ functions into $C^k$ diffeomorphisms;
2. $D_a F$ maps $C^k$ functions into $C^k$ vector fields;
3. $D_a F^{-1}$ maps $C^k$ vector fields into $C^{k-1}$ functions.

Therefore

- $D_a F^{-1}$ “loses derivatives” $\iff$ The inverse of $D_a F$ is unbounded.
- We have to look to map $F$ as a map between Fréchet spaces.
- We need to apply the Nash–Moser Implicit Function Theorem.
Tame Spaces

Stated in terms of Tame Spaces and Tame Maps (Sergeraert 1970)

Definition (Graded Fréchet space)

A Fréchet space $F$ with a family of seminorms $\left\{\| \cdot \|_n\right\}_{n \in \mathbb{N}}$ s.t.

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \ldots$$

The space $C^\infty(B)$ is a graded Fréchet with the family

$$\|f\|_n = \sup_{1 \leq k \leq n} \sup_{x \in B} |f^{(k)}(x)|.$$ 

Spaces of smooth functions are something more:

- $C^\infty(B)$ and $\text{Vec}M$ are Tame Spaces;
- $\text{Diff}_0(M)$ is a Tame Manifold.

Tame space means "scale of Banach spaces".
Tame Maps

Definition (Tame Estimates)

Let $X$ and $Y$ tame spaces and $F : U \subset X \to Y$. $F$ satisfies tame estimates of degree $r$ and base $b$ if there exists $C = C(n)$ such that

$$\|F(a)\|_n \leq C(\|a\|_{n+r} + 1),$$

for every $n \geq b$, $a \in U$.

Definition (Tame Map)

A map $F : U \subset X \to Y$ is a smooth tame map if it is differentiable and together with its differential satisfies tame estimates in a neighborhood of each point.

Example

The map $\text{Exp} : \text{Vec}M \to \text{Diff}(M)$ that sends $f \mapsto e^f$ is a tame map.
Hamilton’s version of Nash–Moser Theorem

**Theorem (Nash–Moser)**

Let $X$ and $Y$ be tame spaces and

$$ F : U \subset X \to Y $$

a smooth tame map. If

- $D_aF(\xi) = \eta$ has a solution for every $a \in U$ and for every $\eta$;
- $DF^{-1} : O \times Y \to X$ is a smooth tame map.

Then $F$ is locally surjective. Moreover in a neighborhood of any point $F$ has a smooth tame right inverse.

The method is:

- prove that $F$ is tame;
- prove that $D_aF(\xi)$ is tame both in $a \in O$ and $\xi \in X$;
- invert $DF$ not only in one point, but in all the neighborhood $U$;
- prove that $(D_aF)^{-1}$ is tame;
Open Problems

We have that for $u_1(t), \ldots, u_\nu(t)$ piecewise constant

\[ F(a) = \exp \int_0^1 \sum_{i=1}^\nu u_i(t) a_i f_j \, dt , \]

is locally surjective. Consider the truncated fourier series of $u_i(t)$, say $u_i^k(t)$. Is the map

\[ F_k(a) = \exp \int_0^1 \sum_{i=1}^\nu u_i^k(t) a_i f_j \, dt , \]

locally surjective too?

- No fixed point argument applies;
- Nash–Moser method (Newton iteration scheme) is the right tool.
“Of course the problem is hard! But this is SISSA... not a small mediocre university!”

“Certo che il problema è difficile! Ma questa è la SISSA... mica una piccola università mediocra!”
Main Theorem

Let $M$ be a compact connected manifold and $\mathcal{F} \subset \text{Vec}M$. If $\text{Gr}\mathcal{F}$ acts transitively on $M$, then there exist

- a neighborhood $\mathcal{O}$ of the identity in $\text{Diff}_0(M)$;
- a positive integer $\mu$

such that every $P \in \mathcal{O}$ can be presented in the form

$$P = e^{a_1f_1} \circ \ldots \circ e^{a_\mu f_\mu},$$

for some $f_1, \ldots, f_\mu \in \mathcal{F}$ and $a_1, \ldots, a_\mu \in C^\infty(M)$. 