Controllability in projection of the simple spectrum bilinear Schrödinger equation

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Abstract: We consider the bilinear Schrödinger equation with several controls and simple-spectrum drift. Under some regularity assumptions on the control operators and generic conditions on the controllability of the Galerkin approximations we show exact controllability in projection on the first n given eigenstates, $n \in \mathbb{N}$ arbitrary. Our methods rely on Lie-algebraic control techniques applied to the Galerkin approximations coupled with classical topological arguments issuing from degree theory.

Keywords: Control of quantum and Schrödinger systems, semigroup and operator theory, control of partial differential equations.

1. INTRODUCTION

In this paper we study the controllability problem for the multi-input bilinear Schrödinger equation

$$i\frac{d\psi}{dt}(t) = (H_0 + u_1(t)H_1 + \ldots + u_p(t)H_p)\psi(t) \qquad (1)$$

where $H_0, \ldots, H_p$ are self-adjoint operators on a Hilbert space $\mathcal{H}$ and the drift Schrödinger operator H_0 (the *internal Hamiltonian*) has simple spectrum. The control functions $u_1(\cdot), \ldots, u_p(\cdot)$, representing external fields, are real-valued and $\psi(\cdot)$ takes values in the unit sphere of $\mathcal{H}$.

In recent years there has been an increasing interest in studying the controllability of the bilinear Schrödinger equation (1), mainly due to its importance for many advanced applications such as Nuclear Magnetic Resonance, laser spectroscopy, and quantum information science. The problem concerns the existence of control laws $(u_1, \ldots, u_p)$ steering the system from a given initial state to a preassigned final state in a given time.

The controllability of system (1) is a well-established topic when the state space $\mathcal{H}$ is finite-dimensional (see for instance D'Alessandro [2008] and reference therein), thanks to general controllability methods for left-invariant control systems on compact Lie groups (Jurdjevic and Sussmann [1972], Jurdjevic and Kupka [1981], Gauthier and Bornard [1982], El Assoudi et al. [1996]).

Considerable efforts have been made to study this problem when $\mathcal{H}$ is infinite-dimensional. When the control operators $H_1, \ldots, H_p$ are bounded, it is known that the bilinear Schrödinger equation is not exactly controllable (see Ball et al. [1982], Turinici [2000]). Hence, one has to look for weaker controllability properties as, for instance, approximate controllability or controllability between eigenstates of the Schödinger operator (which are the most relevant cases from the physical viewpoint). In certain cases, when $\mathcal{H}$ is a function space on a subset of $\mathbb{R}$, a description of reachable sets has been provided (see Beauchard and Coron [2006], Beauchard and Laurent [2010]). In $\mathbb{R}^d$, $d > 1$, or for more general situations, the exact description of the reachable set seems a difficult task and at the moment only approximate controllability results are available. Most of them have been proved in the single-input case (see, in particular, Chambrion et al. [2009], Mirrahimi [2009], Nersesyan [2009, 2010], Boscain et al. [2012a], Nersesyan and Nersisyan [2012]).

Multi-input controllability results have been obtained for specific systems Ervedoza and Puel [2009], Bloch et al. [2010] and some general approximate controllability results between eigenfunctions have been proved via adiabatic methods Boscain et al. [2012b]. The first general multi-input result using Lie-algebraic methods is given in Boscain et al. [2014] where the authors present a sufficient condition for controllability, called Lie–Galerkin Control Condition (see Definition 5 below) of the discrete-spectrum bilinear Schrödinger equation which applies even when the spectrum of the internal Hamiltonian H_0 is very degenerate. The results fully exploit the presence of more than one control and extend to simultaneous controllability, approximate controllability in H^s, and tracking.

Roughly speaking, both the sufficient condition proposed in Boscain et al. [2012a] and the Lie–Galerkin Control Condition are based on the idea of driving the system with control laws that are in resonance with spectral gaps of the internal Hamiltonian H_0 (see also Chambrion [2012]). However, while in Boscain et al. [2012a] the only actions on the system obtained by resonance that are exploited for the controllability are those corresponding to elementary transitions between two eigenstates, no such a restriction is imposed in the Lie–Galerkin Control Condition.

The Lie–Galerkin Control Condition ensures strong controllability properties for the Galerkin approximations: it provides controllability for a fixed Galerkin approximation while avoiding the transfer of population to higher energy levels for higher-order Galerkin approximations. This yields estimates on the difference between the dynamics of the finite-dimensional Galerkin approximation and the original infinite-dimensional system. This fact combined with the continuity of the input-output mapping (see assumption ($\mathbb{A}4$) below) and a topological degree argument ensures exact controllability in projection. More precisely, our main result, Theorem 6, states that given a Hilbert basis $(\phi_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of A, for every given $\mathfrak{n} \in \mathbb{N}$, initial condition $\psi_{\rm in} \in \mathcal{H}$ with $\|\psi_{\rm in}\| = 1$, and final condition $\psi_{\rm f} \in \mathcal{H}$ such that $\|\psi_{\rm f}\| = 1$ with $\langle\psi_{\rm f}, \phi_j\rangle > 0$ for some $j > \mathfrak{n}$ there exists a piecewise constant control $t \mapsto (u_1(t), \dots, u_p(t))$ such that the associated solution $t \mapsto \psi(t)$ of (1) with $\psi(0) = \psi_{\rm in}$ satisfies $\langle \Upsilon^u_T(\psi_{\rm in}), \phi_j\rangle = \langle\psi_{\rm f}, \phi_j\rangle$ for every $j = 1, \dots, \mathfrak{n}$. The result guarantees, for instance, that given any initial condition $\psi_{\rm in}$ and any $n \in \mathbb{N}$, it is possible to steer in finite time $\psi_{\rm in}$ to the orthogonal to $\mathrm{span}\{\phi_1, \dots, \phi_n\}$.

The hypothesis that the final condition $\psi_{\rm f}$ satisfies $\langle\psi_{\rm f}, \phi_j\rangle > 0$ for some $j > \mathfrak{n}$ comes from the fact that, in general, one cannot expect exact controllability *tout court*. Exact controllability is known to be impossible, for instance, when the control operators $H_1, \dots, H_p$ are bounded. Roughly speaking, the regularity of the control potentials, and as a consequence of the input-output mapping, is an obstruction for the exact controllability while, on the other hand, continuity of the input-output mapping is an assumption needed for the application of the topological degree methods used in the proof of Theorem 6 below. In this sense the controllability in projection seems as the strongest general controllability property that one may expect in the framework of bounded control potentials.

2. FRAMEWORK AND MAIN RESULT

Let $p \in \mathbb{N}$, $\delta > 0$, and set $U = [0, \delta]^p$.

Definition 1. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and $A, B_1, \dots, B_p$ be (possibly unbounded) skew-adjoint operators on $\mathcal{H}$, with domains $D(A), D(B_1), \dots, D(B_p)$. Let us introduce the controlled equation

$$\frac{d\psi}{dt}(t) = (A+u_1(t)B_1+\cdots+u_p(t)B_p)\psi(t), \quad u(t) \in U. \quad (2)$$

We say that A satisfies ($\mathbb{A}1$) if the following assumption is true:

($\mathbb{A}1$) A has simple eigenvalues $(i\lambda_k)_{k\in\mathbb{N}}$.

Denote by Φ a Hilbert basis $(\phi_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of A associated with the family of eigenvalues $(i\lambda_k)_{k\in\mathbb{N}}$ and let $\mathcal{L}$ be the set of finite linear combinations of eigenstates, that is, $\mathcal{L} = \bigcup_{k\in\mathbb{N}} \mathrm{span}\{\phi_1, \dots, \phi_k\}$.

We consider the following assumptions:

($\mathbb{A}2$) $\phi_k \in D(B_j)$ for every $k \in \mathbb{N}, j = 1, \dots, p$;
($\mathbb{A}3$) $A + u_1B_1 + \cdots + u_pB_p : \mathcal{L} \to \mathcal{H}$ is essentially skew-adjoint for every $u \in U$.

When $(A, B_1, \dots, B_p, U, \Phi)$ satisfies ($\mathbb{A}1$) $-$ ($\mathbb{A}2$) $-$ ($\mathbb{A}3$) we define the solution of (2) as follows.

Definition 2. The *solution* of (2) with initial condition $\psi_0 \in \mathcal{H}$ associated with a p-uple of piecewise constant controls $u(\cdot) = (u_1(\cdot), \dots, u_p(\cdot))$ is

$$\psi(t) = \Upsilon^u_t(\psi_0),$$

where $[0, T] \ni t \mapsto \Upsilon^u_t \in \mathbf{U}(\mathcal{H})$ is the *propagator* of (2) that associates, with every t in $[0, T]$, the unitary operator

$$\Upsilon^u_t = e^{(t-\sum_{l=1}^{j-1} t_l)(A+u_1^{(j)}B_1+\cdots+u_p^{(j)}B_p)} \circ \cdots$$
$$\cdots \circ e^{t_1(A+u_1^{(1)}B_1+\cdots+u_p^{(1)}B_p)},$$

where $\sum_{l=1}^{j-1} t_l \le t < \sum_{l=1}^{j} t_l$ and $u(\tau) = (u_1^{(j)}, \dots, u_p^{(j)}) \in U$ if $\sum_{l=1}^{j-1} t_l \le \tau < \sum_{l=1}^{j} t_l$.

From now on $\mathcal{U}$ denotes the set of piecewise constant controls with values in U.

We say that $(A, B_1, \dots, B_p, U, \Phi)$ satisfies ($\mathbb{A}$) if it satisfies ($\mathbb{A}1$) $-$ ($\mathbb{A}2$) $-$ ($\mathbb{A}3$) and the following:

($\mathbb{A}4$) The input-output mapping is continuous in the sense that if $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ and $u \in \mathcal{U}$ such that $u_n \to u$ in $L^1([0, T])$ as $n \to \infty$ then $\Upsilon^{u_n}_t\phi$ tends to $\Upsilon^u_t\phi$ in $\mathcal{H}$ uniformly with respect to $t \in [0, T]$ as $n \to \infty$ for every $\phi \in \mathcal{H}$.

Remark 3. In the case in which $B_1, \dots, B_p$ are bounded operators, assumptions ($\mathbb{A}2$), ($\mathbb{A}3$) are clearly verified. Assumption ($\mathbb{A}4$) is the consequence of [Ball et al., 1982, Theorem 3.6]. More general conditions on $B_1, \dots, B_p$ ensuring that ($\mathbb{A}4$) holds true can be found for instance in [Boussaïd et al., 2016, Section 2.3].

For $n \in \mathbb{N}$ we denote by Π_n the orthogonal projection of $\mathcal{H}$ on the span of the first n eigenvectors of A. When it does not create ambiguities we identify $\mathrm{Im}(\Pi_n) = \mathrm{span}\{\phi_1, \dots, \phi_n\}$ with $\mathbb{C}^n$. Given a linear operator Q on $\mathcal{H}$ such that $\mathrm{span}\{\phi_1, \dots, \phi_n\} \subset D(Q)$ we identify the linear operator $\pi_nQ\pi_n$ with its $n \times n$ complex matrix representation with respect to the basis $(\phi_1, \dots, \phi_n)$. We define, for $j = 1, \dots, p$, $A^{(n)} = \Pi_nA\Pi_n$ and $B_j^{(n)} = \Pi_nB_j\Pi_n$.

Definition 4. Let $n \in \mathbb{N}$. The *Galerkin approximation* of (2) of order n is the control system in $\mathbb{C}^n$ described by the equation

$$\dot{x} = \left(A^{(n)} + u_1B_1^{(n)} + \cdots + u_pB_p^{(n)}\right)x.$$

Let us introduce the set Σ_n of spectral gaps associated with the n-dimensional Galerkin approximation as

$$\Sigma_n = \{|\lambda_l - \lambda_k| \mid l, k = 1, \dots, n\}.$$

For every $\sigma \ge 0$, every $m \in \mathbb{N}$, and every $m \times m$ matrix M, let

$$\mathcal{E}_\sigma(M) = (M_{l,k}\delta_{\sigma,|\lambda_l-\lambda_k|})_{l,k=1}^m,$$

where $\delta_{\cdot,\cdot}$ denotes the Kronecker symbol. The $n\times n$ matrix $\mathcal{E}_\sigma(B_j^{(n)})$, $j = 1, \dots, p$, corresponds then to the "selection" in $B_j^{(n)}$ of the spectral gap σ: every element is 0 except the (l, k)-elements such that $|\lambda_l - \lambda_k| = \sigma$.

Define

$$\Xi_n = \{(\sigma, j) \in \Sigma_n \times \{1, \dots, p\} \mid (B_j)_{k,l}\delta_{\sigma,|\lambda_l-\lambda_k|} = 0 \text{ for } k = 1, \dots, n \text{ and } l > n\}. \quad (3)$$

The set Ξ_n can be seen as follows: If $(\sigma, j) \in \Xi_n$ then the matrix $M = \mathcal{E}_\sigma(B_j^{(n)})$ is such that

$$\mathcal{E}_\sigma(B_j^{(N)}) = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & * \end{array}\right) \text{ for every } N > n.$$

The matrix M corresponds to "compatible dynamics" for the n-dimensional Galerkin approximation (*compatible*, that is, with higher-dimensional Galerkin approximations).

Definition 5. (Boscain et al. [2014]). Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy $(\mathbb{A})$. For every $n \in \mathbb{N}$ define

$$\mathcal{M}_n = \left\{A^{(n)}\right\} \cup \left\{\mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n\right\}.$$

We say that the *Lie–Galerkin Control Condition* holds if for every $n_0 \in \mathbb{N}$ there exists $n > n_0$ such that

$$\mathrm{Lie}\mathcal{M}_n \supseteq \mathfrak{su}(n).$$

Recall that $\mathrm{Lie}\mathcal{M}_n$ denotes the Lie subalgebra of $n \times n$ skew-hermitian matrices generated by the elements in $\mathcal{M}_n$ and that $\mathfrak{su}(n)$ is the Lie algebra of all $n \times n$ skew-hermitian traceless matrices.

Our main result is the following.

Theorem 6. Assume that the Lie–Galerkin Control Condition holds. Then for every

- $\mathfrak{n} \in \mathbb{N}$,
- initial condition $\psi_{\mathrm{in}} \in \mathcal{H}$ with $\|\psi_{\mathrm{in}}\| = 1$,
- final condition $\psi_{\mathrm{f}} \in \mathcal{H}$ such that $\|\psi_{\mathrm{f}}\| = 1$ and $\|\Pi_{\mathfrak{n}}(\psi_{\mathrm{f}})\| < 1$,

there exists a piecewise constant control $u : [0, T] \to U$ such that

$$\Pi_{\mathfrak{n}}(\Upsilon_T^u(\psi_{\mathrm{in}})) = \Pi_{\mathfrak{n}}(\psi_{\mathrm{f}}).$$

Let us recall the relation between the Lie–Galerkin Control Condition and approximate controllability of (2).

Definition 7. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy $(\mathbb{A}1)$, $(\mathbb{A}2)$, $(\mathbb{A}3)$. We say that (2) is *approximately controllable* if for every ψ_0, ψ_1 in the unit sphere of $\mathcal{H}$ and every $\varepsilon > 0$ there exists a piecewise constant control function $u : [0, T] \to U$ such that $\|\psi_1 - \Upsilon_T^u(\psi_0)\| < \varepsilon$.

Theorem 8. (Theorem 2.6 in Boscain et al. [2014]). Assume that $(\mathbb{A}1)$, $(\mathbb{A}2)$, $(\mathbb{A}3)$ hold true. If the Lie–Galerkin Control Condition holds then the system

$$\dot{x} = (A + u_1 B_1 + \cdots + u_p B_p)x, \quad u \in U,$$

is approximately controllable.

3. PROOF OF THEOREM 6

3.1 STEP 1: Time reparameterization and interaction framework

In this section we introduce an auxiliary control system whose solutions are, up to phases, trajectories of (2). The exact controllability in projection of (2) will be derived in what follows from the exact controllability in projection of such an auxiliary system, proved in Section 3.3.

For $\omega, v_1, \dots, v_p \in \mathbb{R}$ set $\Theta(\omega, v_1, \dots, v_p) = e^{-\omega A}(v_1 B_1 + \cdots + v_p B_p)e^{\omega A} : \mathcal{L} \to \mathcal{H}$. Note that

$$\begin{aligned}\Theta(\omega, v_1, \dots, v_p)_{jk} &= \langle \phi_k, \Theta(\omega, v_1, \dots, v_p)\phi_j \rangle \\ &= e^{i(\lambda_k - \lambda_j)\omega}(v_1 B_1 + \cdots + v_p B_p)_{jk}.\end{aligned}$$

Denote by $\mathcal{V}$ the set of triples of functions $(\alpha, v, \omega) : [0, \infty) \to \{0, 1\} \times U \times [0, \infty)$ such that α, v are piecewise constants and ω is continuous and piecewise affine with $\dot{\omega} \geq 1$ almost everywhere.

Consider the system

$$\dot{y}(t) = (\alpha(t)A + \Theta(\omega(t), v_1(t), \dots, v_p(t)))y(t), \qquad (4)$$

for (α, v, ω) in $\mathcal{V}$. Admissibile solutions of (4) are absolutely continuous functions $y : [0, T] \to \mathcal{H}$ satisfying

$$\frac{d}{dt}\langle \phi_n, y(t) \rangle = -\langle (\alpha(t)A + \Theta(\omega(t), v_1(t), \dots, v_p(t)))\phi_n, y(t) \rangle,$$

for any $n \in \mathbb{N}$ and for almost every $t \in [0, T]$. We denote the propagator of (4) associated with the triple (α, v, ω) as $t \mapsto \mathcal{T}_t(\alpha, v, \omega)$.

Lemma 9. For every $(\alpha, v, \omega) \in \mathcal{V}$ and every initial condition $y_0 \in \mathcal{H}$ there exists an admissible solution $y : [0, \infty) \to \mathcal{H}$ of (4) associated with (α, v, ω) starting from y_0. Moreover, there exist $u \in \mathcal{U}$ and $s : [0, \infty) \to [0, \infty)$ strictly increasing such that for every $T > 0$

$$\Upsilon_{s(T)}^u(y(0)) = e^{\omega(T)A}y(T). \qquad (5)$$

Proof. Let (α, v, ω) be in $\mathcal{V}$ and assume that $y : [0, \infty) \to \mathcal{H}$ is an admissible solution of (4) associated with (α, v, ω). The function $\psi(t) = e^{\omega(t)A}y(t)$ satisfies, for every $n \in \mathbb{N}$ and for almost every $t \in [0, \infty)$,

$$\frac{d}{dt}\langle \phi_n, \psi \rangle = -\langle ((\dot{\omega} + \alpha)A + v_1 B_1 + \cdots + v_p B_p)\phi_n, \psi \rangle.$$

Define the time rescaling

$$s(t) = \int_0^t (\alpha(\tau) + \dot{\omega}(\tau))\, d\tau, \qquad \text{for every } t \geq 0.$$

The function $s : [0, \infty) \to [0, \infty)$ is strictly increasing since $\alpha + \dot{\omega} \geq 1$ almost everywhere on $[0, \infty)$. Consider the piecewise constant function,

$$u(t) = \frac{1}{\alpha(s^{-1}(t)) + \dot{\omega}(s^{-1}(t))} v(s^{-1}(t)), \quad t \in [0, \infty).$$

Notice that u takes values in U, since $\alpha + \dot{\omega} \geq 1$.

Both $t \mapsto \tilde{\psi}(t) = \psi \circ s^{-1}(t)$ and the solution of (2) associated with u satisfy

$$\frac{d}{dt}\langle \phi_n, \varphi(t) \rangle = -\langle (A + u_1(t)B_1 + \cdots + u_p(t)B_p)\phi_n, \varphi(t) \rangle, \qquad (6)$$

for $n \in \mathbb{N}$ and for almost every $t \in [0, T']$. Finally notice that the norm of the solutions of the system of equations (6) ($n \in \mathbb{N}$) is constant. In particular this gives uniqueness of the solution for a given initial datum. It also show that for every initial condition $y_0 \in \mathcal{H}$ there exists an admissible solution of (4) associated with (α, v, ω) starting from y_0.

3.2 STEP 2: Phase tuning

Lemma 9 can be used together with Lemma 10 below, in order to replace system (2) by system (4) in the proof of Theorem 6. The idea is to correct the dephasing term $e^{\mu A}$ in (5) by letting the system evolve freely for a suitable amount τ of time. For more details on how Lemma 10 is applied, see Section 3.4.

Lemma 10. Let $\psi \in \mathcal{H}$, $\mu > 0$, $N \in \mathbb{N}$, and $\mathcal{N}$ be a neighborhood of $\Pi_N(e^{\mu A}\psi)$ in $\mathrm{span}\{\phi_1, \dots, \phi_N\}$. Then there exists $\tau \geq 0$ such that $e^{\tau A}\mathcal{N}$ is a neighborhood of $\Pi_N\psi$ in $\mathrm{span}\{\phi_1, \dots, \phi_N\}$.

Proof. Without loss of generality we can assume that $\mathcal{N}$ is a $2N$-dimensional open ball $B_{2\varepsilon}(\Pi_N(e^{\mu A}\psi))$ of radius 2ε for some $\varepsilon > 0$, centered at $\Pi_N(e^{\mu A}\psi)$.

Consider the sequence $(\beta_k)_{k\in\mathbb{N}}$ with $\beta_k = B_\varepsilon(\Pi_N(e^{k\mu A}\psi))$. All the elements of $(\beta_k)_k$ are of constant volume and are contained in the subset $B_{\|\Pi_N(\psi)\|+\varepsilon}(0)$ of $\mathrm{span}\{\phi_1,\dots,\phi_N\}$. Since the latter has finite volume, there exist $\ell, m \in \mathbb{N}$ such that $\beta_\ell \cap \beta_{\ell+m} \neq \emptyset$. Since $e^{-\ell\mu A}$ is an isomorphism of $\mathrm{span}\{\phi_1,\dots,\phi_N\}$ we deduce that

$$B_\varepsilon(\Pi_N(\psi)) \cap B_\varepsilon(\Pi_N(e^{m\mu A}\psi)) \neq \emptyset.$$

Thus $\Pi_N(\psi) \in B_{2\varepsilon}(\Pi_N(e^{m\mu A}\psi))$. Therefore

$$\Pi_N(\psi) \in B_{2\varepsilon}(\Pi_N(e^{m\mu A}\psi)) = e^{(m-1)\mu A}\mathcal{N}.$$

This proves the lemma with $\tau = (m-1)\mu$.

3.3 STEP 3: Normal controllability

In this section we prove, that neglecting the phase, the Galerkin approximations of system (4) are normally controllable.

Let n_0 be as in the statement of Theorem 6 and let $n > n_0$ as given by the Lie–Galerkin Control Condition. Define the collection of matrices

$$\mathcal{W}_n = \left\{A^{(n)}\right\} \cup \left\{\mathcal{E}_0(B_j^{(n)}) \mid j \in \{1,\dots,p\}\right\}$$
$$\cup \left\{\mathcal{E}_0(B_j^{(n)}) + \nu\mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n \text{ and } \sigma \neq 0\right\},$$

where Ξ_n is defined as in (3) and $\nu = \prod_{k=2}^\infty \cos\left(\frac{\pi}{2k}\right) > 0$.

Consider the auxiliary control system

$$\dot{x} = M(t)x, \quad M(t) \in \mathcal{W}_n,$$

where M plays the role of control.

Let $\tilde{\psi}_\mathrm{f} \in S^{2n-1}$ be such that $\Pi_{n_0}(\psi_\mathrm{f}) = \Pi_{n_0}(\tilde{\psi}_\mathrm{f})$.

Notice that $\mathrm{Lie}(\mathcal{W}_n) = \mathrm{Lie}(\mathcal{M}_n)$ and, by the Lie–Galerkin Control Condition, $\mathrm{Lie}(\mathcal{W}_n) \supseteq \mathfrak{su}(n)$. By classical results in control (see Jurdjevic and Sussmann [1972] and [Sussmann, 1976, Theorem 4.3]) there exist $M_1,\dots,M_k \in \mathcal{W}_n$ and $t_1,\dots,t_k > 0$ such that the map

$$E : (s_1,\dots,s_k) \mapsto e^{s_k M_k} \circ \cdots \circ e^{s_1 M_1}(\phi_1)$$

has rank $2n-1$ at $(t_1,\dots,t_k)$ and

$$E(t_1,\dots,t_k) = \tilde{\psi}_\mathrm{f}.$$

Since $n > n_0$ and $\|\Pi_{n_0}(\tilde{\psi}_\mathrm{f})\| < 1$, there exist $j_1,\dots,j_{2n_0} \in \{1,\dots,k\}$ such that the map

$$F : (s_{j_1},\dots,s_{j_{2n_0}}) \mapsto$$
$$\Pi_{n_0}(E(t_1,\dots,t_{j_1-1},s_{j_1},\dots,t_{j_{2n_0}-1},s_{j_{2n_0}},\dots,t_k)) \quad (7)$$

has rank $2n_0$ at $(t_{j_1},\dots,t_{j_{2n_0}})$ and

$$F(t_{j_1},\dots,t_{j_{2n_0}}) = \Pi_{n_0}(\psi_\mathrm{f}) \quad \left(= \Pi_{n_0}(\tilde{\psi}_\mathrm{f})\right).$$

Now, let $\rho > 0$ be such that

$$X := B_\rho(t_{j_1},\dots,t_{j_{2n_0}})$$

is compactly contained in $(0,+\infty)^{2n_0}$ and F is a diffeomorphism between X and $F(X)$. Let

$$\eta = \inf_{(s_1,\dots,s_{2n_0})\in\partial X} \|F(s_1,\dots,s_{2n_0}) - \Pi_{n_0}(\psi_\mathrm{f})\| > 0.$$

Set $T_s = t_1 + \cdots + t_{j_1-1} + s_{j_1} + t_{j_1+1} + \cdots + t_{j_{2n_0}-1} + s_{j_{2n_0}} + t_{j_{2n_0}+1} + \cdots + t_k$.

Lemma 11. For every $\eta > 0$ there exists a map

$$\begin{array}{ccc} \bar{X} & \to & \mathcal{V} \\ (s_1,\dots,s_{2n_0}) & \mapsto & (\alpha^s(\cdot), v^s(\cdot), \omega^s(\cdot)) \end{array}$$

such that $\omega^s(T_s)$ does not depend on s, the mapping

$$\tilde{G} : (s_1,\dots,s_{n_0}) \mapsto \Pi_{n_0}\left(\mathcal{T}_{T_s}(\alpha^s, v^s, \omega^s)\phi_1\right)$$

is continuous on $\bar{X}$, and

$$\max_{s\in\bar{X}} \|F(s) - \tilde{G}(s)\| < \eta.$$

The proof of Lemma 11, given in Appendix A, is rather technical and follows the arguments in [Boscain et al., 2014, Lemma 4.4 and Proposition 4.5].

Lemma 12. There exist $w \in \mathcal{U}$ and $T > 0$ such that the image $G(X)$ of the mapping

$$\begin{array}{cccc} G : & X & \to & \mathrm{span}\{\phi_1,\dots,\phi_{n_0}\} \\ & (s_1,\dots,s_{n_0}) & \mapsto & \Pi_{n_0}\left(\mathcal{T}_{T_s}(\alpha^s, v^s, \omega^s)\Upsilon_T^w(\psi_\mathrm{in})\right) \end{array}$$

contains $\Pi_{n_0}(\psi_\mathrm{f})$ in its interior.

Proof. By Theorem 8, as a consequence of the Lie–Galerkin Control Condition, we have that system (2) is approximately controllable by means of controls in $\mathcal{U}$. Therefore for every $\varepsilon > 0$ there exist $u_\varepsilon \in \mathcal{U}$ and $T > 0$ such that $\|\phi_1 - \Upsilon_T^{u_\varepsilon}(\psi_\mathrm{in})\| < \varepsilon$.

Define the mapping

$$\begin{array}{cccc} G : & X & \to & \mathrm{span}\{\phi_1,\dots,\phi_{n_0}\} \\ & (s_1,\dots,s_{n_0}) & \mapsto & \Pi_{n_0}\left(\mathcal{T}_{T_s}(\alpha^s, v^s, \omega^s)\Upsilon_T^{u_\varepsilon}(\psi_\mathrm{in})\right). \end{array}$$

Note that since $\|\Pi_{n_0}\mathcal{T}_{T_s}(\alpha^s, v^s, \omega^s)\| \leq 1$ for every $s \in \bar{X}$ then

$$\max_{s\in\bar{X}} \|G(s) - \tilde{G}(s)\| \leq \|\phi_1 - \Upsilon_T^{u_\varepsilon}(\psi_\mathrm{in})\| < \varepsilon.$$

Hence, for ε sufficiently small, setting $w = u_\varepsilon$,

$$\max_{s\in\partial X} \|G(s) - F(s)\| < \eta.$$

By Lemma 13 in Appendix A, $\Pi_{n_0}(\psi_\mathrm{f}) \in \mathrm{int}(G(X))$.

3.4 STEP 4: Final step

We are now ready to prove Theorem 6.

Proof of Theorem 6. Lemma 12 guarantees the existence of a neighborhood $\mathcal{N} \subset G(X)$ of $\Pi_{n_0}(\psi_2)$. Namely, for every $\zeta \in \mathcal{N}(\subset \mathrm{span}\{\phi_1,\dots,\phi_{n_0}\})$ there exists $s \in X$ such that

$$\zeta = \Pi_{n_0}\left(\mathcal{T}_{T_s}(\alpha^s, v^s, \omega^s)\Upsilon_T^w(\psi_\mathrm{in})\right).$$

Recall that by Lemma 11 there exists $\mu > 0$ such that $\omega^s(T_s) = \mu$ for every $s \in X$.

By Lemma 9, for every $\zeta \in \mathcal{N}$ there exist $T'_\zeta > 0$ and $u_\zeta \in \mathcal{U}$ such that

$$\Pi_{n_0}(\Upsilon_{T'_\zeta}^{u_\zeta}(\psi_1)) = e^{\mu A}\zeta = \Pi_{n_0}(e^{\mu A}\zeta).$$

Applying Lemma 10 with $N = n_0$ and $\psi = \psi_2$, we deduce that there exists τ such that, setting

$$w_\zeta(t) = \begin{cases} w(t) & t \in [0,T), \\ u_\zeta(t-T) & t \in [T, T+T'_\zeta), \\ 0 & t \geq T + T'_\zeta, \end{cases} \qquad \zeta \in \mathcal{N},$$

we have that

$$\left\{\Pi_{n_0}(\Upsilon_{T+T'_\zeta+\tau}^{w_\zeta})(\psi_1) \mid \zeta \in \mathcal{N}\right\},$$

is a neighborhood of $\Pi_{n_0}(\psi_2)$. Moreover $w_\zeta \in \mathcal{U}$ for every $\zeta \in \mathcal{N}$. This concludes the proof of Theorem 6.

Appendix A. TECHNICAL LEMMATA

A.1 A useful topological tool

The following topological result is standard in degree theory. We provide its proof for completeness.

Lemma 13. Let $X \subset \mathbb{R}^n$ be open and bounded and let $F \in C(\overline{X}, \mathbb{R}^n)$ be a diffeomorphism between X and $F(X)$. Assume that y_0 is in $F(X)$ and consider $0 < \varepsilon \leq \mathrm{dist}(y_0, F(\partial X))$. If $G \in C(\overline{X}, \mathbb{R}^n)$ satisfies

$$\max_{x \in \partial X} |F(x) - G(x)| < \varepsilon$$

then $y_0 \in \mathrm{int}(G(X))$.

Proof. Consider the homotopy $h \in C([0,1] \times \bar{X}, \mathbb{R}^n)$ defined by $h(t,x) = tF(x) + (1-t)G(x)$. By definition of ε we have that $0 \neq h(t,x)$ for every $t \in [0,1]$ and $x \in \partial X$. In particular the topological degree $\mathrm{d}(h(t,\cdot), X, y_0)$ is well defined for every $t \in [0,1]$. Since $F|_X$ is a diffeomorphism and $y_0 \in F(X)$ then $\mathrm{d}(F, X, y_0) \neq 0$ and by homotopy invariance $\mathrm{d}(G, X, y_0) \neq 0$. Hence $y_0 \in \mathrm{int}(G(X))$.

A.2 Convexification

The following technical result has been proved in [Boscain et al., 2012a, Lemma 4.3].

Lemma 14. Let κ be a positive integer and $\gamma_1, \ldots, \gamma_\kappa \in \mathbb{R} \setminus \{0\}$ be such that $|\gamma_1| \neq |\gamma_j|$ for $j = 2, \ldots, \kappa$. Let

$$\varphi(t) = (e^{it\gamma_1}, \ldots, e^{it\gamma_\kappa}).$$

Then, for every $\tau_0 \in \mathbb{R}$, we have

$$\overline{\mathrm{conv}\varphi([\tau_0, \infty))} \supseteq \nu S^1 \times \{(0, \ldots, 0)\},$$

where

$$\nu = \prod_{k=2}^{\infty} \cos\left(\frac{\pi}{2k}\right) > 0.$$

Moreover, for every $R > 0$ and $\xi \in S^1$ there exists a sequence $(\tau_k)_{k=1}^{\infty}$ such that $\tau_1 \geq \tau_0$, $\tau_{k+1} - \tau_k > R$, and

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \varphi(\tau_k) = (\nu\xi, 0, \ldots, 0).$$

A.3 Piecewise construction of the control in Lemma 11

Lemma 11 is proved by concatenating the controls given by Lemma 15 below. In this section n_0 is as in the statement of Theorem 6 and $n > n_0$ is given by the Lie–Galerkin Control Condition as in Section 3.3.

Lemma 15. For every $\eta_0 > 0$, $M \in \mathcal{W}_n$, ω_0, and $r \in [a,b]$ for $0 < a \leq b$, there exist

- $\mu > \omega_0$,
- $\alpha \in \{0, 1\}$, and
- $v \in U$,

not depending on r, and

- $\omega_r : [0,b] \to [\omega_0, +\infty)$ continuous piecewise affine with $\dot{\omega}_r \geq 1$,

such that

(i) $\|e^{tM} - \Pi_n \mathcal{T}_t(\alpha, v, \omega_r)\Pi_n\| \leq \eta_0$, for every $t \in [0,r]$,
(ii) $\omega_r(0) = \omega_0$ and $\omega_r(r) = \mu$,
(iii) the map $r \mapsto \omega_r$ is continuous from $[a,b]$ to $W^{1,1}([0,b])$.

Proof. For every $N \geq n$ consider the *Galerkin approximation* of (4) of order N, that is the system

$$\dot{x} = (\alpha A^{(N)} + \Theta^{(N)}(\omega, v_1, \ldots, v_p))x, \quad x \in \mathbb{C}^N, \quad \text{(A.1)}$$

where $\Theta^{(N)}(\omega, v_1, \ldots, v_p) = \pi_N \Theta(\omega, v_1, \ldots, v_p)\pi_N$.

STEP 1. Fix $N \geq n$. Then, for every $h \in (0,a)$ and $r \in [a,b]$, we construct a triple (α, v, ω_r^h) such that for every $h \in (0,a)$ $r \mapsto \omega_r^h$ verifies *(ii)* and *(iii)* and, moreover, for every $r \in [a,b]$, the family of flows $t \mapsto \Phi_{r,t}^h$ associated with the non-autonomous vector fields $\alpha A^{(N)} + \Theta^{(N)}(\omega_r^h, v_1, \ldots, v_p)$ satisfies

$$\|\Pi_n \Phi_{r,t}^h \Pi_n - e^{tM}\| \to 0 \quad \text{(A.2)}$$

as h goes to 0, uniformly for $t \in [0,r]$. Depending on M we have three cases: either $M = A^{(n)}$, $M = \mathcal{E}_0(B_j^{(n)}) + \nu\mathcal{E}_\sigma(B_j^{(n)})$ for some j and $\sigma \neq 0$ such that $(\sigma, j) \in \Xi_n$, or $M = \mathcal{E}_0(B_j^{(n)})$ for some $j \in \{1, \ldots, p\}$.

Case 1. If $M = A^{(n)}$ then $\alpha = 1$ and $v_1 = \cdots = v_p = 0$. Let $\mu > b + \omega_0$ and for every $h \in (0,a)$ consider the function ω_r^h equal to $\omega_0 + t$ for $t \in [0, r-h]$ and $\omega_r^h(t) = \omega_0 + r - h + (t - r + h)(\mu - r + h)/h$ for $t \in (r-h, b]$. Notice that for every $h \in (0,a)$ and for every μ the function ω_r^h is continuous and piecewise affine with $\dot{\omega}_r^h \geq 1$. Clearly the convergence (A.2) holds true.

Case 2. If $M = \mathcal{E}_0(B_j^{(n)}) + \nu\mathcal{E}_\sigma(B_j^{(n)})$ for some j, then take such a j minimal and set $v_j = 1$ and $\alpha = v_k = 0$ for $k \neq j$. Call $\hat{M} = \mathcal{E}_0(B_j^{(N)}) + \nu\mathcal{E}_\sigma(B_j^{(N)})$.

Fix any $N \geq n$. Apply Lemma 14 with $\gamma_1 = \sigma$, $\{\gamma_2, \ldots, \gamma_\kappa\} = \Sigma_N \setminus \{\sigma\}$, $\xi = 1$, $R = b$, and $\tau_0 = \omega_0$. Then there exists a sequence $(\tau_k)_{k=1}^{\infty}$ such that $\tau_1 \geq \omega_0$, $\tau_{k+1} - \tau_k > b$, and

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} (e^{i\tau_k\gamma_1}, \ldots, e^{i\tau_k\gamma_\kappa}) = (\nu, 0, \ldots, 0).$$

In particular, for every $h > 0$ there exists K such that

$$\left| \frac{1}{K} \sum_{k=1}^{K} e^{i(\lambda_l - \lambda_m)\tau_k} b_{ml}^{(j)} - (\hat{M})_{m,l} \right| < h$$

for every $1 \leq l, m \leq N$.

Consider a partition of $[0,r)$ in K intervals I_k of equal length $|I_k| = r/K$. By a simple smoothing procedure (see for instance [Boscain et al., 2012a, Proposition 5.5]) one can construct a continuous piecewise affine approximation $\omega_r^h : [0,r] \to \mathbb{R}$ of the piecewise function $t \mapsto \sum_{k=1}^{K} \tau_k \chi_{I_k}(t)$ with $\dot{\omega}_h \geq 1$ almost everywhere and such that $\omega_r^h(r) = \tau_{K+1}$ and

$$\left\| \int_0^t (\Theta^{(N)}(\omega_r^h(\tau), v_1(\tau), \ldots, v_p(\tau)))d\tau - t\hat{M} \right\| \to 0$$

uniformly with respect to $t \in [0,r]$ as h tends to 0. Then extend ω_r^h on $(r,b]$ with $\omega_r^h(t) = \tau_{K+1} + t - r$ for $t \in [r,b]$. As a consequence and thanks to [Agrachev and Sachkov, 2004, Lemma 8.2], we have

$$\left\| \Phi_{r,t}^h - e^{t\hat{M}} \right\| \to 0$$

uniformly with respect to $t \in [0,r]$ as h tends to 0. Notice that $\omega_r^h(r) = \tau_{K+1} =: \mu$ does not depend on r (but may depend on h).

Case 3. If $M = \mathcal{E}_0(B_j^{(n)})$ the same argument of Case 2 can be carried out by applying Lemma 14 with γ_1 in $(0,\infty)\backslash\Sigma_N$, $\{\gamma_2,\ldots,\gamma_\kappa\} = \Sigma_N$, $\xi = 1$, $R = b$, and $\tau_0 = \omega_0$.

STEP 2. We can now consider N sufficiently large in such a way the difference between the propagators of (A.1) and (4) is small. The proof is given in [Boscain et al., 2014, Proposition 4.5] and it is omitted. Then choose h sufficiently small depending on N and η_0. This concludes the proof of Lemma 15.

Proof of Lemma 11. The proof is based on an iterative application of Lemma 15 for every $\ell = 1,\ldots,k$ in (7). For every $s \in X$ and $\ell = 1,\ldots,k$ write $\tau_\ell = t_\ell$ if $\ell \notin \{j_1,\ldots,j_{2n_0}\}$ and $\tau_\ell = s_{j_\ell}$ otherwise.

First, for $\ell = 1$ apply Lemma 15 with $\omega_0 = 0$ and $a = b = t_1$ if $1 \notin \{j_1,\ldots,j_{2n_0}\}$ or $[a,b] = [t_1-\delta, t_1+\delta]$ otherwise. We define $\alpha^s = \alpha$, $v^s = v$, and $\omega^s = \omega_{\tau_1}$ on $[0,\tau_1]$ for $\tau_1 \in [a,b]$.

For $\ell > 1$ apply Lemma 15 with $\omega_0 = \omega^s(\tau_1+\cdots+\tau_{\ell-1})$ and $a = b = t_\ell$ if $\ell \notin \{j_1,\ldots,j_{2n_0}\}$ or $[a,b] = [t_\ell-\delta, t_\ell+\delta]$ otherwise. Again, we define $\alpha^s = \alpha$, $v^s = v$, and $\omega^s = \omega_{\tau_\ell}$ on $[\tau_1+\cdots+\tau_{\ell-1}, \tau_1+\cdots+\tau_\ell]$ for $\tau_\ell \in [a,b]$.

For η_0 sufficiently small one has the desired statement. The continuity of

$$s \mapsto \mathcal{T}_{T_s}(\alpha^s, v^s, \omega^s)$$

follows from (iii) in Lemma 15 and Assumption $(\mathbb{A}4)$.

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