Exact controllability in projections of the bilinear Schrödinger equation*

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Abstract

We consider the bilinear Schrödinger equation with discrete-spectrum drift. We show, for $n \in \mathbb{N}$ arbitrary, exact controllability in projections on the first n given eigenstates. The controllability result relies on a generic controllability hypothesis on some associated finite-dimensional approximations. The method is based on Lie-algebraic control techniques applied to the finite-dimensional approximations coupled with classical topological arguments issuing from degree theory.

1 Introduction

In this paper we study the controllability problem for the multi-input Schrödinger equation

$$i\frac{d\psi}{dt}(t) = (H_0 + u_1(t)H_1 + \ldots + u_p(t)H_p)\psi(t) \tag{1}$$

where $H_0, \ldots, H_p$ are self-adjoint operators on a Hilbert space $\mathcal{H}$ and the drift Schrödinger operator H_0 (the *internal Hamiltonian*) has discrete spectrum. The control functions $u_1(\cdot), \ldots, u_p(\cdot)$, representing external fields, are real-valued and $\psi(\cdot)$ takes values in the unit sphere of $\mathcal{H}$.

In recent years there has been an increasing interest in studying the controllability of the bilinear Schrödinger equation (1), mainly due to its importance for many applications such as laser spectroscopy or quantum information. The problem concerns the design of control laws $(u_1, \ldots, u_p)$ steering the system from a given initial state to a pre-assigned final state in a given time.

The controllability of system (1) is a well-established topic when the state space $\mathcal{H}$ is finite-dimensional (see for instance [14] and reference therein), thanks to general controllability methods for left-invariant control systems on compact Lie groups ([18, 17, 16, 15]).

We are interested here in the case in which $\mathcal{H}$ is infinite-dimensional. When the control operators $H_1, \ldots, H_p$ are bounded, it is known that the bilinear Schrödinger equation is not exactly controllable (see [3, 28]). Hence, it is natural to look for weaker controllability properties such as approximate controllability or controllability between eigenstates of the Schödinger operator. In certain cases, when $\mathcal{H}$ is a function space on a a real interval, a description of reachable sets has been provided (see [4, 5, 21]). In $\mathbb{R}^d$, $d > 1$, the exact description of the reachable set seems a difficult task. However,

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approximate controllability results have been obtained with different techniques: adiabatic control ([1, 10]), Lyapunov methods ([20, 22, 23, 24]), and Lie-algebraic methods ([13, 7, 8, 12, 9, 6, 19, 25]).

The Lie-algebraic approach developed in [13, 8, 12, 9] serves as basis for the analysis in this paper. The basic idea is to drive the system with control laws that are in resonance with spectral gaps of the internal Hamiltonian H_0. The resonances are used to idenfify finite-dimensional dynamics which can be tracked with arbitrary precision by the infinite-dimensional system. In [9] we introduced the Lie–Galerkin Control Condition (see Definition 4 below), which ensures that the family of finite-dimensional dynamics obtained in this way is controllable. The condition applies also for very degenerate spectra of the internal Hamiltonian H_0 and guarantees operator controllability, approximate controllability in finer topologies, and tracking.

In this paper we go beyond approximate controllability and prove that the Lie–Galerkin Control Condition implies a stronger controllability property: exact controllability in projections. The Lie–Galerkin Control Condition provides controllability for a fixed finite-dimensional approximation while avoiding the transfer of population to higher energy levels for higher-dimensional approximations. This yields estimates on the difference between the dynamics of the finite-dimensional approximation and the original infinite-dimensional system. This fact combined with the continuity of the input-output mapping (see Assumption ($\mathbb{A}5$)) and a topological degree argument ensures exact controllability in projections. More precisely, our main result, Theorem 3, states that given a Hilbert basis $(\phi_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of A, for every given $\mathfrak{n} \in \mathbb{N}$, initial condition $\psi_{\rm in} \in \mathcal{H}$ with $\|\psi_{\rm in}\| = 1$, and final condition $\psi_{\rm f} \in \mathcal{H}$ such that $\|\psi_{\rm f}\| = 1$ with $\langle \psi_{\rm f}, \phi_j\rangle > 0$ for some $j > \mathfrak{n}$ there exists an admissible control $t \mapsto (u_1(t), \dots, u_p(t))$ such that the associated solution $t \mapsto \psi(t)$ of (1) with $\psi(0) = \psi_{\rm in}$ satisfies $\langle \Upsilon^u_T(\psi_{\rm in}), \phi_j\rangle = \langle \psi_{\rm f}, \phi_j\rangle$ for every $j = 1, \dots, \mathfrak{n}$. The result guarantees, for instance, that given any initial condition $\psi_{\rm in}$ and any $n \in \mathbb{N}$, it is possible to steer in finite time $\psi_{\rm in}$ to the orthogonal complement of $\mathrm{span}\{\phi_1, \dots, \phi_n\}$.

The hypothesis that the final condition $\psi_{\rm f}$ satisfies $\langle \psi_{\rm f}, \phi_j\rangle > 0$ for some $j > \mathfrak{n}$ cannot be removed. Since, as we have already recalled, one cannot expect exact controllability *tout court* if the control operators $H_1, \dots, H_p$ are regular (e.g., continuous). The regularity of the control operators, and as a consequence of the input-output mapping, is therefore an obstruction for the exact controllability while, on the other hand, continuity of the input-output mapping is an assumption needed for the application of the topological degree methods used in the proof of Theorem 3 below. In this sense the controllability in projections is the strongest general controllability property that one may expect in the framework of bounded control potentials.

2 Framework and main result

Let $p \in \mathbb{N}$, $\delta > 0$, and $U = U_1 \times \cdots \times U_p$ with either $U_j = [0, \delta]$ or $U_j = [-\delta, \delta]$. For simplicity of notation we consider the bilinear control systems obtained by replacing the operators in (1) by $A = -iH_0$ and $B_j = -iH_j$, $j = 1, \dots, p$. This leads to the following definition.

Definition 1. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and $A, B_1, \dots, B_p$ be (possibly unbounded) skew-adjoint operators on $\mathcal{H}$, with domains $D(A), D(B_1), \dots, D(B_p)$. Let us introduce the controlled equation

$$\frac{d\psi}{dt}(t) = (A + u_1(t)B_1 + \cdots + u_p(t)B_p)\psi(t), \quad u(t) \in U. \tag{2}$$

We say that A satisfies ($\mathbb{A}1$) if the following assumption holds true.

($\mathbb{A}1$) A has discrete spectrum with infinitely many distinct eigenvalues (possibly degenerate).

Note that ($\mathbb{A}1$) is true whenever A has compact resolvent. Denote by Φ a Hilbert basis $(\phi_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of A associated with the family of eigenvalues $(i\lambda_k)_{k\in\mathbb{N}}$ and let $\mathcal{L}$ be the set of finite linear combinations of eigenstates, that is,

$$\mathcal{L} = \bigcup_{k\in\mathbb{N}} \operatorname{span}\{\phi_1, \ldots, \phi_k\}.$$

We consider the following assumptions:

($\mathbb{A}2$) $\phi_k \in D(B_j)$ for every $k \in \mathbb{N}, j = 1, \ldots, p$;

($\mathbb{A}3$) $A + u_1 B_1 + \cdots + u_p B_p : \mathcal{L} \to \mathcal{H}$ is essentially skew-adjoint for every $u \in U$.

When $(A, B_1, \ldots, B_p, \Phi, U)$ satisfies $(\mathbb{A}1) - (\mathbb{A}2) - (\mathbb{A}3)$ we define the solution of (2) as follows.

Definition 2. We say that $u \in L^\infty([0,T], \mathbb{R}^p)$ is *admissible* for (2) if $u(t) \in U$ for almost every $t \in [0,T]$ and, for every $\psi_0 \in \mathcal{H}$, there exists $\psi : [0,T] \to \mathcal{H}$ such that $\psi(0) = \psi_0$ the function $t \mapsto \langle \psi(t), \phi_k\rangle$ is absolutely continuous for every $k \in \mathbb{N}$ and satisfies

$$\frac{d}{dt}\langle \phi_k, \psi(t)\rangle = -\langle (A + u_1(t)B_1 + \cdots + u_p(t)B_p)\phi_k, \psi(t)\rangle, \tag{3}$$

for almost every $t \in [0,T]$. The function $t \mapsto \psi(t)$ is called *solution* of (2) with initial condition $\psi_0 \in \mathcal{H}$ associated with the control u.

Assumption ($\mathbb{A}3$) implies that the norm of the solutions given by Definition 2 is constant along the evolution. In particular it guarantees the uniqueness of solutions. We can therefore define the unitary propagator of (2), denoted by Υ^u_t, as follows.

Definition 3. Let $u : [0,T] \to U$ be admissible for (2). The mapping $[0,T] \ni t \mapsto \Upsilon^u_t$ where $\Upsilon^u_t \psi_0$ is the evaluation at time t of the solution of (2) with initial condition $\psi_0 \in \mathcal{H}$ associated with u, is called *propagator* of (2).

Let $(A, B_1, \ldots, B_p, \Phi, U)$ satisfy $(\mathbb{A}1) - (\mathbb{A}2) - (\mathbb{A}3)$ and let $\mathcal{U} \subset L^\infty([0,\infty), U)$. We say that $(A, B_1, \ldots, B_p, \Phi, U, \mathcal{U})$ satisfies ($\mathbb{A}4$) if

($\mathbb{A}4$) every $u \in \mathcal{U}$ is admissible.

Assumption ($\mathbb{A}4$) holds true for the class of piecewise constant controls and, under suitable regularity conditions, for the class of smooth controls as detailed in the following two remarks.

Remark 1. Let $(A, B_1, \ldots, B_p, \Phi, U)$ satisfy $(\mathbb{A}1) - (\mathbb{A}2) - (\mathbb{A}3)$ and let $u(\cdot) = (u_1(\cdot), \ldots, u_p(\cdot))$ be a p-tuple of piecewise constant controls on $[0,T]$ with value in U. Then u is admissible for (2) and the propagator is given by

$$\Upsilon^u_t = e^{(t - \sum_{l=1}^{j-1} t_l)(A + u_1^{(j)} B_1 + \cdots + u_p^{(j)} B_p)} \circ \cdots \circ e^{t_1 (A + u_1^{(1)} B_1 + \cdots + u_p^{(1)} B_p)},$$

where $\sum_{l=1}^{j-1} t_l \le t < \sum_{l=1}^{j} t_l$ and $u(\tau) = (u_1^{(j)}, \ldots, u_p^{(j)}) \in U$ if $\sum_{l=1}^{j-1} t_l \le \tau < \sum_{l=1}^{j} t_l$. Indeed, since

$$\langle \phi_n, e^{t(A + u_1 B_1 + \cdots + u_p B_p)}\psi_0\rangle = \langle e^{-t(A + u_1 B_1 + \cdots + u_p B_p)}\phi_n, \psi_0\rangle\,,$$

then $t \mapsto \psi(t) = \Upsilon^u_t \psi_0$ satisfies (3) for almost every $t \in [0,T]$.

Remark 2. Let $u \in C^1([0,T],U)$ and $B_1,\dots,B_p$ be A-bounded with A-bound smaller than $1/\delta$, namely

- $D(B_j) \supset D(A)$,
- there exists $a < 1/\delta$ and $b \in \mathbb{R}$ such that for all $\phi \in D(A)$ one has
$$\|B_j\phi\| \leq a\|A\phi\| + b\|\phi\|,$$

for every $j = 1,\dots,p$. Then by the Kato–Rellich theorem ([26, Theorem X.12]) and [26, Theorem X.70] u is admissible for (2).

We say that $(A, B_1,\dots,B_p,\Phi,U,\mathcal{U})$ satisfies $(\mathbb{A})$ if it satisfies $(\mathbb{A}1)-(\mathbb{A}2)-(\mathbb{A}3)-(\mathbb{A}4)$ and the following additional assumption.

($\mathbb{A}5$) The input-output mapping is continuous in the sense that if $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ and $u \in \mathcal{U}$ are such that $u_n \to u$ in $L^1([0,T])$ as $n \to \infty$ then $\Upsilon_t^{u_n}\phi$ tends to $\Upsilon_t^u\phi$ in $\mathcal{H}$ uniformly with respect to $t \in [0,T]$ as $n \to \infty$ for every $\phi \in \mathcal{H}$.

Remark 3. In the case in which A satisfies $(\mathbb{A}1)$ and $B_1,\dots,B_p$ are bounded operators, assumptions $(\mathbb{A}2)$, $(\mathbb{A}3)$ are clearly verified. Assumption $(\mathbb{A}5)$ is the consequence of [3, Theorem 3.6]. More general conditions on $B_1,\dots,B_p$ ensuring that $(\mathbb{A}5)$ holds true can be found for instance in [11, Section 2.3].

For $n \in \mathbb{N}$ we denote by Π_n the projection of $\mathcal{H}$ on the span of the first n eigenvectors of A, namely

$$\begin{array}{rccc} \Pi_n: & \mathcal{H} & \to & \mathcal{H} \\ & \psi & \mapsto & \sum_{k=1}^n \langle\phi_k,\psi\rangle\phi_k. \end{array}$$

When it does not create ambiguities we identify $\mathrm{Im}(\Pi_n) = \mathrm{span}\{\phi_1,\dots,\phi_n\}$ with $\mathbb{C}^n$. Given a linear operator Q on $\mathcal{H}$ we identify the linear operator $\pi_n Q \pi_n$ preserving $\mathrm{span}\{\phi_1,\dots,\phi_n\}$ with its $n\times n$ complex matrix representation with respect to the basis $(\phi_1,\dots,\phi_n)$. We define

$$A^{(n)} = \Pi_n A \Pi_n \quad \text{and} \quad B_j^{(n)} = \Pi_n B_j \Pi_n,$$

for every $j = 1,\dots,p$.

Let us introduce the set Σ_n of spectral gaps associated with the first n eigenvalues of A as

$$\Sigma_n = \{|\lambda_l - \lambda_k| \mid l,k = 1,\dots,n\}.$$

For every $\sigma \geq 0$, every $m \in \mathbb{N}$, and every $m\times m$ matrix M, let

$$\mathcal{E}_\sigma(M) = (M_{l,k}\delta_{\sigma,|\lambda_l-\lambda_k|})_{l,k=1}^m,$$

where $\delta_{\cdot,\cdot}$ denotes the Kronecker symbol. The $n\times n$ matrix $\mathcal{E}_\sigma(B_j^{(n)})$, $j=1,\dots,p$, corresponds then to the "selection" in $B_j^{(n)}$ of the spectral gap σ: every element is 0 except the (l,k)-elements such that $|\lambda_l - \lambda_k| = \sigma$.

Define

$$\Xi_n = \left\{(\sigma,j) \in \Sigma_n \times \{1,\dots,p\} \mid (B_j)_{k,l}\delta_{\sigma,|\lambda_l-\lambda_k|} = 0, \text{ for every } k = 1,\dots,n \text{ and } l > n\right\}. \qquad (4)$$

The set Ξ_n can be seen as follows: If $(\sigma,j) \in \Xi_n$ then the matrix $M = \mathcal{E}_\sigma(B_j^{(n)})$ is such that

$$\mathcal{E}_\sigma(B_j^{(N)}) = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & * \end{array}\right) \text{ for every } N > n.$$

Definition 4 ([9])**.** For every $n \in \mathbb{N}$ define

$$\mathcal{M}_n = \left\{A^{(n)}\right\} \cup \left\{\mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n \text{ and } j \text{ is such that } (0,j) \in \Xi_n\right\}$$
$$\cup \left\{\mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n, \sigma \neq 0, U_j = [-\delta, \delta]\right\}.$$

We say that the *Lie–Galerkin Control Condition* holds if for every $n_0 \in \mathbb{N}$ there exists $n > n_0$ such that

$$\mathrm{Lie}\mathcal{M}_n \supseteq \mathfrak{su}(n).$$

The Lie–Galerkin Control Condition is a sufficient condition for approximate controllability as stated in Theorem 1 below.

Definition 5. Let $(A, B_1, \ldots, B_p, \Phi, U, \mathcal{U})$ satisfy $(\mathbb{A}1) - (\mathbb{A}2) - (\mathbb{A}3) - (\mathbb{A}4)$. We say that (2) is *approximately controllable* by means of controls in $\mathcal{U}$ if for every ψ_0, ψ_1 in the unit sphere of $\mathcal{H}$ and every $\varepsilon > 0$ there exists $u \in \mathcal{U}$ and $T > 0$ such that $\|\psi_1 - \Upsilon_T^u(\psi_0)\| < \varepsilon$.

Theorem 1 ([9, Theorem 2.6])**.** *Let $(A, B_1, \ldots, B_p, \Phi, U)$ satisfy $(\mathbb{A}1) - (\mathbb{A}2) - (\mathbb{A}3)$. If the Lie–Galerkin Control Condition holds then system* (2) *is approximately controllable by means of piecewise constant controls.*

Theorem 1 can be extended to the class of smooth controls as stated in Theorem 2 below. The proof of this result is given in Section 4.

Theorem 2. *Let $\mathcal{U}$ be the set of C^∞ functions with values in U and $(A, B_1, \ldots, B_p, \Phi, U, \mathcal{U})$ satisfy* $(\mathbb{A}1)-(\mathbb{A}2)-(\mathbb{A}3)-(\mathbb{A}4)$*. If the Lie–Galerkin Control Condition holds then system* (2) *is approximately controllable by means of controls in $\mathcal{U}$.*

Our main result is the approximate controllability in projections as stated below. Theorem 2 is one of the main tools in its proof.

Theorem 3. *Let $\mathcal{U}$ be either the set of piecewise constant functions with values in U or the set of C^∞ functions with values in U. Let $(A, B_1, \ldots, B_p, \Phi, U, \mathcal{U})$ satisfy $(\mathbb{A})$. Assume that the Lie–Galerkin Control Condition holds. Then for every $\mathfrak{n} \in \mathbb{N}$, $\varepsilon > 0$, for every initial condition $\psi_{\mathrm{in}} \in \mathcal{H}$ with $\|\psi_{\mathrm{in}}\| = 1$ and every final condition $\psi_{\mathrm{f}} \in \mathcal{H}$ such that $\|\psi_{\mathrm{f}}\| = 1$ and $\|\Pi_{\mathfrak{n}}(\psi_{\mathrm{f}})\| < 1$, there exists $u : [0, T] \to U$, $u \in \mathcal{U}$ such that*

$$\Pi_{\mathfrak{n}}(\Upsilon_T^u(\psi_{\mathrm{in}})) = \Pi_{\mathfrak{n}}(\psi_{\mathrm{f}}) \qquad \textit{and} \qquad \|\Upsilon_T^u(\psi_{\mathrm{in}}) - \psi_{\mathrm{f}}\| < \varepsilon.$$

3 Finite-dimensional approximations of infinite-dimensional propagators

In this section we introduce an auxiliary control system whose solutions are, up to phases, trajectories of (2), as showed in Lemma 4 below.

For $t, u_1, \ldots, u_p \in \mathbb{R}$ set $\Theta(t, u) = \Theta(t, u_1, \ldots, u_p) = e^{-tA}(u_1 B_1 + \cdots + u_p B_p)e^{tA} : \mathcal{L} \to \mathcal{H}$. Note that

$$\Theta(t, u_1, \ldots, u_p)_{jk} = \langle \phi_k, \Theta(t, u_1, \ldots, u_p)\phi_j \rangle = e^{i(\lambda_k - \lambda_j)t}\left(u_1 (B_1)_{jk} + \cdots + u_p (B_p)_{jk}\right).$$

Consider the nonautonomous control system

$$\dot{y}(t) = \Theta(t, u_1(t), \ldots, u_p(t))y(t). \tag{5}$$

Admissibile solutions of (5) are, as in Definition 2, absolutely continuous functions $y : [0,T] \to \mathcal{H}$ satisfying

$$\frac{d}{dt}\langle \phi_n, y(t)\rangle = -\langle \Theta(t, u_1(t), \ldots, u_p(t))\phi_n, y(t)\rangle, \tag{6}$$

for any $n \in \mathbb{N}$ and for almost every $t \in [0,T]$.

Lemma 4. *Let $u \in \mathcal{U}$. Then u is admissible for* (5) *and the corresponding admissible solution $y : [0,T] \to \mathcal{H}$ associated with the initial condition $y(0)$ satisfies*

$$\Upsilon^u_t(y(0)) = e^{tA}y(t),$$

for every $t \in [0,T]$.

Proof. The function

$$y(t) = e^{-tA}\Upsilon^u_t(y(0))$$

satisfies

$$\begin{aligned}
\frac{d}{dt}\langle \phi_n, y(t)\rangle &= \frac{d}{dt}e^{it\lambda_n}\langle \phi_n, \Upsilon^u_t(y(0))\rangle \\
&= i\lambda_n e^{it\lambda_n}\langle \phi_n, \Upsilon^u_t(y(0))\rangle + e^{it\lambda_n}\frac{d}{dt}\langle \phi_n, \Upsilon^u_t(y(0))\rangle \\
&= i\lambda_n e^{it\lambda_n}\langle \phi_n, \Upsilon^u_t(y(0))\rangle - e^{it\lambda_n}\langle (A + u_1(t)B_1 + \cdots + u_p(t)B_p)\phi_n, \Upsilon^u_t(y(0))\rangle \\
&= -e^{it\lambda_n}\langle (u_1(t)B_1 + \cdots + u_p(t)B_p)\phi_n, e^{tA}y(t)\rangle \\
&= -\langle \Theta(t, u_1(t), \ldots, u_p(t))\phi_n, y(t)\rangle,
\end{aligned}$$

for every $n \in \mathbb{N}$ and for almost every $t \in [0,T]$. □

The following lemma is inspired by [12, Theorem 1].

Lemma 5. *Let $n \in \mathbb{N}$, $\sigma > 0$, $j \in \{1, \ldots, p\}$, ν_0 and ν_1 in $\mathbb{R}$, and $a, b \in \mathbb{R}$ such that $0 < a < b$. For every $N > n$ let $\omega^N = (\underbrace{0, \ldots, 0}_{j-1}, v^N, \underbrace{0, \ldots, 0}_{p-j}) : \mathbb{R} \to \mathbb{R}^p$ be periodic of period $T = 2\pi/\sigma$, such that*

$$\int_0^T v^N(t)dt = \nu_0, \qquad \int_0^T v^N(t)e^{i\sigma t}dt = \nu_1, \tag{7}$$

and

$$\int_0^T v^N(t)e^{im\sigma t}dt = 0, \qquad \textit{for every } m \geq 2 \textit{ such that } m\sigma \in \Sigma_N. \tag{8}$$

Assume that one of the following three conditions is satisfied: (i) (σ, j) and $(0, j)$ are in Ξ_n; (ii) $(\sigma, j) \in \Xi_n$ and $\nu_0 = 0$; (iii) $(0, j) \in \Xi_n$ and $\nu_1 = 0$. Assume, moreover, that ω^N/K is admissible for every $N \in \mathbb{N}$ and every $K \in \mathbb{N}$ large enough. Then

$$\lim_{N\to\infty}\lim_{K\to\infty}\|\Upsilon^{\tau\omega^N/K}_{KT} - e^{KTA}\exp(\tau(\nu_0\mathcal{E}_0(B^{(n)}_j) + \nu_1\mathcal{E}_\sigma(B^{(n)}_j)))\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} = 0,$$

uniformly with respect to $\tau \in [a,b]$, where $\exp(\tau(\nu_0\mathcal{E}_0(B^{(n)}_j) + \nu_1\mathcal{E}_\sigma(B^{(n)}_j))) : \mathbb{C}^n \to \mathbb{C}^n$ is identified with an operator on $\mathcal{H}$.

Proof. **Step 1.** Fix, for now, $N > n$ and consider the Galerkin approximation of order N of system (5), namely the system associated with $\Theta^{(N)}(t,u) = \Pi_N \Theta(t,u)\Pi_N$. Then, for every $s \le t$ and $\tau \in [a,b]$,

$$\overrightarrow{\exp} \int_{Ks}^{Kt} \Theta^{(N)}\left(r, \frac{\tau\omega^N(r)}{K}\right) dr = \overrightarrow{\exp} \int_s^t \Theta^{(N)}(Kr, \tau\omega^N(Kr))\, dr$$
$$= \overrightarrow{\exp} \int_s^t e^{-rKA^{(N)}} \tau v^N(Kr) B_j^{(N)} e^{rKA^{(N)}}\, dr, \tag{9}$$

where the notation $\overrightarrow{\exp}$ is used to denote the *chronological exponent*, see [2]. We claim that

$$\int_s^t e^{-rKA^{(N)}} v^N(Kr) B_j^{(N)} e^{rKA^{(N)}} dr \to \frac{t-s}{T}\left(\nu_0\mathcal{E}_0(B_j^{(N)}) + \nu_1\mathcal{E}_\sigma(B_j^{(N)})\right) \tag{10}$$

as $K \to \infty$, uniformly with respect to $s, t \in [0,T]$. Indeed, this follows from (8) and the fact that for every $\sigma' \in \mathbb{R} \setminus \sigma\mathbb{Z}$

$$\frac{1}{K}\int_{Ks}^{Kt} v^N(r) e^{i\sigma' r} dr \to 0$$

as $K \to \infty$, as it can be seen by developing v^N in Fourier series.

Hence by (9), (10), and averaging arguments (see, for instance, [2, Lemma 8.2]) we deduce that

$$\overrightarrow{\exp} \int_{Ks}^{Kt} \Theta^{(N)}\left(r, \frac{\tau\omega^N(r)}{K}\right) dr \to \exp\left(\frac{(t-s)\tau}{T}\left(\nu_0\mathcal{E}_0(B_j^{(N)}) + \nu_1\mathcal{E}_\sigma(B_j^{(N)})\right)\right) \tag{11}$$

as $K \to \infty$, uniformly with respect to $s, t \in [0,T]$ and $\tau \in [a,b]$.

Step 2. Let

$$y(t) = e^{-tA}\Upsilon_t^{\tau\omega^N/K}(\psi),$$

for $\psi \in \Pi_n(\mathcal{H})$. By Lemma 4, $y(t)$ is a solution of (5).

By variation of constants formula and since $\Pi_N\Theta(s,u)(\mathrm{I} - \Pi_N)$ is an operator uniformly bounded with respect to $s \in \mathbb{R}$ and $u \in U$, we deduce from (6) that

$$\Pi_n y(t) = \Pi_n\ \overrightarrow{\exp} \int_0^t \Theta^{(N)}\left(s, \frac{\tau\omega^N(s)}{K}\right) ds\,(\psi)$$
$$+ \Pi_n \int_0^t \left(\overrightarrow{\exp} \int_s^t \Theta^{(N)}\left(r, \frac{\tau\omega^N(r)}{K}\right) dr\right) \Pi_N\Theta\left(s, \frac{\tau\omega^N(s)}{K}\right)(\mathrm{I} - \Pi_N) y(s) ds.$$

We are left to prove that $\lim_{N\to\infty}\lim_{K\to\infty} \Pi_n e^{-KTA}\Upsilon_{KT}^{\tau\omega^N/K}\Pi_n = \exp(\nu_0\mathcal{E}_0(B_j^{(n)}) + \nu_1\mathcal{E}_\sigma(B_j^{(n)}))$ uniformly with respect to $\tau \in [a,b]$. Indeed, by unitarity of the evolution of (5), this also implies that

$$\lim_{N\to\infty}\lim_{K\to\infty}(\mathrm{I} - \Pi_n)e^{-KTA}\Upsilon_{KT}^{\tau\omega^N/K}\Pi_n = 0,$$

uniformly with respect to $\tau \in [a,b]$.

By (11), for every $N > n$ we have that

$$\lim_{K\to\infty} \Pi_n\ \overrightarrow{\exp} \int_0^{KT} \Theta^{(N)}\left(s, \frac{\tau\omega^N(s)}{K}\right) ds\ \Pi_n = \exp(\tau(\nu_0\mathcal{E}_0(B_j^{(n)}) + \nu_1\mathcal{E}_\sigma(B_j^{(n)}))),$$

uniformly with respect to $\tau \in [a,b]$. Hence, we are left to prove that for every $\varepsilon > 0$ and every N large enough one has

$$\limsup_{K\to\infty} \|\Pi_n \int_0^{KT} \left(\overrightarrow{\exp} \int_s^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) \Pi_N \Theta \left(s, \frac{\tau\omega^N(s)}{K} \right) (\mathrm{I}-\Pi_N) e^{-sA} \Upsilon_s^{\tau\omega^N/K} ds \Pi_n \| < \varepsilon,$$

uniformly with respect to $\tau \in [a,b]$. Notice that

$$\int_0^{KT} \left(\overrightarrow{\exp} \int_s^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) \Pi_N \Theta \left(s, \frac{\tau\omega^N(s)}{K} \right) (\mathrm{I}-\Pi_N) e^{-sA} \Upsilon_s^{\tau\omega^N/K} ds = \\ \int_0^{T} \left(\overrightarrow{\exp} \int_{Ks}^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) \Pi_N \Theta \left(Ks, \tau\omega^N(Ks) \right) (\mathrm{I}-\Pi_N) e^{-KsA} \Upsilon_{Ks}^{\tau\omega^N/K} ds.$$

We are going prove, therefore, that for every $\varepsilon > 0$ and every N large enough

$$\limsup_{K\to\infty} \|\Pi_n \left(\overrightarrow{\exp} \int_{Ks}^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) \Pi_N \Theta \left(Ks, \tau\omega^N(Ks) \right) (\mathrm{I}-\Pi_N) \| < \varepsilon,$$

uniformly with respect to $s \in [0,T]$ and $\tau \in [a,b]$. We have that

$$\begin{aligned} &\Pi_n \left(\overrightarrow{\exp} \int_{Ks}^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) \Pi_N \Theta \left(Ks, \tau\omega^N(Ks) \right) (\mathrm{I}-\Pi_N) \\ &\quad = \Pi_n \left(\overrightarrow{\exp} \int_{Ks}^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) \Pi_n \Theta \left(Ks, \tau\omega^N(Ks) \right) (\mathrm{I}-\Pi_N) \\ &\qquad + \Pi_n \left(\overrightarrow{\exp} \int_{Ks}^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) (\Pi_N - \Pi_n) \Theta \left(Ks, \tau\omega^N(Ks) \right) (\mathrm{I}-\Pi_N). \end{aligned} \tag{12}$$

Fix N large enough so that

$$\|\Pi_n \Theta(s,u)(\mathrm{I}-\Pi_N)\| < \frac{\varepsilon}{T}$$

for every $s \in \mathbb{R}$ and $u \in U$. Concerning the term in (12), by assumption ($\mathbb{A}2$) one has that

$$(\Pi_N - \Pi_n)\Theta(s,u)(\mathrm{I}-\Pi_N)$$

is a bounded operator on $\mathcal{H}$, uniformly with respect to $s \in \mathbb{R}$ and $u \in U$.

Notice that by the assumptions on σ, j, ν_0, and ν_1 it holds

$$\Pi_n \left(\nu_0 \mathcal{E}_0(B_j^{(N)}) + \nu_1 \mathcal{E}_\sigma(B_j^{(N)}) \right) (\Pi_N - \Pi_n) = 0.$$

Hence, by (11),

$$\Pi_n \left(\overrightarrow{\exp} \int_{Ks}^{KT} \Theta^{(N)} \left(r, \frac{\tau\omega^N(r)}{K} \right) dr \right) (\Pi_N - \Pi_n)$$

tends to 0 as $K \to \infty$ uniformly with respect to $s \in [0,T]$ and $\tau \in [a,b]$. Hence the term in (12) goes to 0 as $K \to \infty$ uniformly with respect to $\tau \in [a,b]$. □

3.1 Efficiency of admissible controls

Let us discuss the values of ν_0 and ν_1 which can be obtained with a control satisfying the hypothesis of Lemma 5. We distinguish two cases depending on the nature of the control set U_j.

If $U_j = [-\delta, \delta]$ then for every $\nu_0, \nu_1 \in \mathbb{R}$ a simple example of periodic function, independent of N, satisfying (7) and (8) is given by

$$v(t) = \frac{\sigma}{2\pi} \left(\nu_0 + 2\nu_1 \cos(\sigma t) \right). \tag{13}$$

Indeed by orthogonality of trigonometric functions one has that

$$\int_0^T v(t) e^{im\sigma t} dt = 0$$

for every integer $m \geq 2$ (in particular for every $m \geq 2$ such that $m\sigma \in \Sigma_N$).

In the case in which $U_j = [0, \delta]$ the restriction on the sign of v^N represents an additional requirement. If $\nu_0 = 0$ then $v^N = 0$. Otherwise, up to replacing v^N by v^N/ν_0, one can assume that $\nu_0 = 1$. The value of $|\nu_1|$ is a measure of the *efficiency* of the control v^N (see [12, Section 2.4]). Notice, for instance, that the efficiency of the functions $\frac{\sigma}{2\pi}(1 \pm \cos(\sigma t))$ is $1/2$. Hence, by convexity, one can choose a v^N, independent of N, so that $\nu_0 = 1$ and ν_1 is any prescribed value in $[-1/2, 1/2]$.

However, in both cases, these simple and explicit choices of v^N are somehow too rigid for our purposes. Indeed these functions are not piecewise constant and, on the other hand, one cannot concatenate smooth functions of the form (13) corresponding to distinct values of σ in a smooth (i.e. C^∞) way. In order to overcome this issue we restrict condition (8) to $m \geq 2$ such that $m\sigma \in \Sigma_N$ allowing v^N to depend on N. This is the rationale for Lemma 6 below, which shows that small perturbations of functions of the form (13) yield admissible controls with efficiency arbitrarily close to $1/2$. In the C^∞ case we further prescribe the approximating admissible controls to be zero with all derivatives at 0 and T so that concatenations of function of this kind are smooth.

Lemma 6. *Let $\sigma > 0$ and define $T = 2\pi/\sigma$. Denote by $\mathscr{U}_T$ either the class of nonnegative piecewise constant functions on $[0, T]$ or the class of nonnegative C^∞ functions on $[0, T]$ that are zero with all derivatives at 0 and T. Then, for every $\nu_1 \in (-1/2, 1/2)$ and $N \in \mathbb{N}$ there exists $w \in \mathscr{U}_T$ such that*

(i) $\int_0^T w(t) dt = 1$,

(ii) $\int_0^T w(t) e^{i\sigma t} dt = \nu_1$,

(iii) $\int_0^T w(t) e^{im\sigma t} dt = 0$ *for every $m \geq 2$ such that $m\sigma \in \Sigma_N$.*

Proof. Let $\varphi_0 \in \{t \mapsto 1 - \cos(\sigma t), t \mapsto 1 + \cos(\sigma t)\}$ and

$$\{\varphi_1, \dots, \varphi_k\} = \{t \mapsto \sin(m\sigma t), t \mapsto \cos(m\sigma t) \mid m \geq 2 \text{ such that } m\sigma \in \Sigma_N\}.$$

Let $\alpha \in [4/5, 1)$. For every $\varepsilon > 0$ consider $w_0, w_1, \dots, w_k \in \mathscr{U}_T$ such that

$$w_0(t) \geq \varepsilon^\alpha \quad \text{for every } t \in [\varepsilon, T - \varepsilon], \tag{14}$$

$$w_0(t) = 0 \quad \text{for every } t \in [0, \varepsilon^2) \cup (T - \varepsilon^2, T]$$

$$w_j(t) = 0 \quad \text{for every } j = 1, \dots, k, \text{ and } t \in [0, \varepsilon) \cup (T - \varepsilon, T],$$

and such that

$$\|w_j - \varphi_j\|_{L^2([0,T])} < C\varepsilon, \qquad \text{for every } j = 0, \dots, k, \tag{15}$$

for some C independent of ε. Indeed, since $\alpha \geq 4/5$ one easily checks that (14) is compatible with (15).

Consider the solution $(c_1, \dots, c_k)$ of the linear system

$$\begin{cases} \langle w_1, \varphi_1\rangle c_1 + \cdots + \langle w_k, \varphi_1\rangle c_k &= -\langle w_0, \varphi_1\rangle, \\ \quad\vdots \qquad \vdots & \qquad\quad \vdots \\ \langle w_1, \varphi_k\rangle c_1 + \cdots + \langle w_k, \varphi_k\rangle c_k &= -\langle w_0, \varphi_k\rangle. \end{cases}$$

Notice that the solution exists since the matrix of the system is close to an invertible diagonal matrix for ε sufficiently small. Moreover $|c_j| \leq C\varepsilon$ for every $j = 1, \dots, k$ (possibly considering a larger constant C).

Then define

$$w = \frac{w_0}{T} + \sum_{j=1}^{k} c_j w_j.$$

The function w belongs to $\mathscr{U}_T$. Indeed since $\alpha < 1$ then $w \geq 0$ for ε small enough.

By construction w satisfies (iii). Possibly rescaling w by a factor $\left(\int_0^T w dt\right)^{-1} = 1 + O(\varepsilon)$ point (i) is satisfied. Notice now that

$$\int_0^T w(t) e^{i\sigma t} dt = \frac{1}{T}\int_0^T \varphi_0(t) e^{i\sigma t} dt + O(\varepsilon) = \pm\frac{1}{2} + O(\varepsilon).$$

The conclusion follows by convexity and letting ε go to zero. □

Remark 4. One could relax the condition on v^N in (8) by replacing the set Σ_N by

$$\{|\lambda_l - \lambda_k| \mid l, k = 1, \dots, N, \text{ and } \langle \phi_l, B_j \phi_k\rangle \neq 0\}.$$

The proof of Lemma 5 remains unchanged since the condition in (8) is only used in (10).

4 Proof of Theorem 2

4.1 Finite-dimensional exact controllability implies infinite-dimensional approximate controllability up to phases

Let $n_0 \in \mathbb{N}$ and $n > n_0$ be given by the Lie–Galerkin Control Condition. Define the collection of matrices

$$\begin{aligned}\mathcal{W}_n = &\left\{A^{(n)}\right\} \cup \left\{\mathcal{E}_0(B_j^{(n)}) \mid (0,j) \in \Xi_n\right\} \\ &\cup \left\{\mathcal{E}_0(B_j^{(n)}) + \nu\mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n \text{ and } \sigma, j \text{ are such that } (0,j) \in \Xi_n, \sigma \neq 0, \nu \in (-1/2, 1/2)\right\} \\ &\cup \left\{\mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n,\ \sigma \neq 0,\ \text{and } U_j = [-\delta, \delta]\right\},\end{aligned}$$

where Ξ_n is defined as in (4). The matrices in $\mathcal{W}_n$ correspond to the asymptotic dynamics obtained in Lemma 5 for the admissible choices of values for ν_0 and ν_1 (see Lemma 6). Notice that $\mathrm{Lie}(\mathcal{W}_n) = \mathrm{Lie}(\mathcal{M}_n)$ and, by the Lie–Galerkin Control Condition, $\mathrm{Lie}(\mathcal{W}_n) \supseteq \mathfrak{su}(n)$.

Consider the auxiliary control system

$$\dot{x} = M(t)x, \quad M(t) \in \mathcal{W}_n, \tag{16}$$

where M plays the role of control.

Proposition 7 below, based on Lemma 5, states that any propagator of (16) can be approximated by a propagator of system (2).

Proposition 7. *Let $n, k \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $0 < a < b$, and $M_1, \dots, M_k \in \mathcal{W}_n$. For every $\varepsilon > 0$ and $\tau_1, \dots, \tau_k \in [a, b]$ there exist $u \in \mathcal{U}$, $T_u > 0$, and $\gamma \geq 0$ such that*

$$\|\Upsilon^u_{T_u} - e^{\gamma A} \circ e^{\tau_k M_k} \circ \cdots \circ e^{\tau_1 M_1}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} < \varepsilon$$

where every $e^{\tau_\ell M_\ell}$ is identified with an operator on $\mathcal{H}$. Moreover, γ can be taken independent of $\tau_1, \dots, \tau_k \in [a, b]$.

More precisely, for $\ell = 1, \dots, k$ and $\tau_1, \dots, \tau_k \in [a, b]$ there exist $T_\ell \geq 0$ and $\omega_\ell : [0, T_\ell] \to \mathbb{R}^p$, such that for K large enough u can be taken as the concatenation

$$u = 0|_{[0,\chi_1]} * \underbrace{\frac{\tau_1\omega_1}{K} * \cdots * \frac{\tau_1\omega_1}{K}}_{K-times} * 0|_{[0,\chi_2]} * \underbrace{\frac{\tau_2\omega_2}{K} * \cdots * \frac{\tau_2\omega_2}{K}}_{K-times} * \cdots * 0|_{[0,\chi_k]} * \underbrace{\frac{\tau_k\omega_k}{K} * \cdots * \frac{\tau_k\omega_k}{K}}_{K-times}$$

with $\chi_\ell \geq 0$ continuously depending on τ_ℓ, where $0|_{[0,\chi_\ell]}$ denotes the function $[0, \chi_\ell] \ni t \mapsto (0, \dots, 0) \in \mathbb{R}^p$.

Proof. We construct u as the concatenation $u^1 * \cdots * u^k$ of k controls $u^\ell : [0, \vartheta_\ell] \to \mathbb{R}^p$, $\ell = 1, \dots, k$, namely, $u(t) = u^\ell(t - (\vartheta_1 + \cdots + \vartheta_{\ell-1}))$ for $t \in [\vartheta_1 + \cdots + \vartheta_{\ell-1}, \vartheta_1 + \cdots + \vartheta_\ell)$, with $u^\ell = 0|_{[0,\chi_\ell]} * \frac{\tau_\ell\omega_\ell}{K} * \cdots * \frac{\tau_\ell\omega_\ell}{K}$ and $\vartheta_\ell = \chi_\ell + KT_\ell$.

If $\mathcal{U}$ is the set of piecewise constant functions with values in U then this concatenation clearly results in an admissible control. If $\mathcal{U}$ is the set of C^∞ functions with values in U then, in order to guarantee that u is admissible, every control u^ℓ is contructed as a smooth function which is zero with all derivatives at 0 and ϑ_ℓ.

Let us construct recursively u^ℓ and ϑ_ℓ for $\ell \in \{1, \dots, k\}$.

- For $\ell = 1$ we have two cases.
  - If $M_1 = A^{(n)}$ then consider $T_1 = 0$ and $\chi_1 = \tau_1$.
  - If M_1 is of the form $\nu_0\mathcal{E}_0(B_j^{(n)}) + \nu_1\mathcal{E}_\sigma(B_j^{(n)})$ for some $\sigma > 0$ and $j \in \{1, \dots, p\}$ then we apply Lemma 5. Therefore there exist $N > n$, a control ω^N and K such that

    $$\|\Upsilon^{\tau_1\omega^N/K}_{KT} - e^{KTA}e^{\tau_1 M_1}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} < \varepsilon/k,$$

    for every $\tau_1 \in [a, b]$, where $T = 2\pi/\sigma$. Then, $T_1 = T$, $\chi_1 = 0$, $\omega_1 = \omega^N|_{[0,T]}$.
- Let $\ell \geq 2$ and assume that, for every $\tau_1, \dots, \tau_{\ell-1} \in [a, b]$, the control u is constructed on $[0, \vartheta_1 + \cdots + \vartheta_\ell)$. Then there exists $\gamma \geq 0$ such that

  $$\|e^{-\gamma A}\Upsilon^u_{\vartheta_1+\cdots+\vartheta_\ell} - e^{\tau_{\ell-1}M_{\ell-1}} \circ \cdots \circ e^{\tau_1 M_1}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} < \varepsilon\frac{\ell - 1}{k}, \tag{17}$$

  for every $\tau_1, \dots, \tau_{\ell-1} \in [a, b]$. Again we distinguish two cases.

- If $M_\ell = A^{(n)}$ then set $T_\ell = 0$ and $\chi_\ell = \tau_\ell$.
- Consider now the case in which M_ℓ is of the form $\nu_0\mathcal{E}_0(B_j^{(n)}) + \nu_1\mathcal{E}_\sigma(B_j^{(n)})$ for some $\sigma > 0$ and $j \in \{1, \ldots, p\}$.

  Take a time $\gamma' \geq 0$ such that $\gamma' + \gamma$ is an integer multiple of $T = 2\pi/\sigma$, for instance $\gamma' = \lceil \frac{\gamma}{T} \rceil T - \gamma$, and let $\chi_\ell = \gamma'$. Apply now Lemma 5. Then there exist $N > n$ a control ω^N and K such that

  $$\|\Upsilon_{KT}^{\tau_\ell\omega^N/K} - e^{KTA}e^{\tau_\ell M_\ell}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} < \varepsilon/k,$$

  for every $\tau_\ell \in [a, b]$.

  Since $\gamma' + \gamma$ is an integer multiple of $T = 2\pi/\sigma$, we have that $[M_\ell, e^{-(\gamma+\gamma')A}] = 0$, which implies that $[e^{\tau_\ell M_\ell}, e^{-(\gamma+\gamma')A}] = 0$. Hence

  $$\begin{aligned}
  &\|e^{-(\gamma+\gamma'+KT)A}\Upsilon^u_{\vartheta_1+\cdots+\vartheta_{\ell+1}} - e^{\tau_\ell M_\ell} \circ \cdots \circ e^{\tau_1 M_1}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} \\
  &\qquad = \|e^{-(\gamma+\gamma'+KT)A}\Upsilon_{KT}^{\tau_\ell\omega^N/K}e^{\gamma' A}\Upsilon^u_{\vartheta_1+\cdots+\vartheta_\ell} - e^{\tau_\ell M_\ell} \circ \cdots \circ e^{\tau_1 M_1}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} \\
  &\qquad \leq \|\Upsilon_{KT}^{\tau_\ell\omega^N/K} - e^{KTA}e^{\tau_\ell M_\ell}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} \\
  &\qquad\quad + \|e^{-(\gamma+\gamma')A}e^{\tau_\ell M_\ell}\Upsilon^u_{\vartheta_1+\cdots+\vartheta_\ell} - e^{\tau_\ell M_\ell} \circ \cdots \circ e^{\tau_1 M_1}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} \\
  &\qquad \leq \frac{\varepsilon}{k} + \|e^{-\gamma A}\Upsilon^u_{\vartheta_1+\cdots+\vartheta_\ell} - e^{\tau_{\ell-1}M_{\ell-1}} \circ \cdots \circ e^{\tau_1 M_1}\|_{L(\Pi_n(\mathcal{H}),\mathcal{H})} \leq \frac{\varepsilon\ell}{k},
  \end{aligned}$$

  which concludes the induction with $T_\ell = T$ and $\omega_\ell = \omega^N|_{[0,T]}$ and with the role of γ in (17) now played by $\gamma + \gamma' + KT$.

This concludes the construction of u with $T_u = \vartheta_1 + \cdots + \vartheta_k$.

Finally, notice that, in the case in which $\mathcal{U}$ is the set of C^∞ functions with values in U, thanks to Lemma 6 at each step we can assume that the control ω_ℓ is zero with all derivatives at 0 and T (and so the control u is admissible). Indeed, if $\nu_0 \neq 0$ then Lemma 6 applies directly, while if $\nu_0 = 0$ then ω_ℓ can be defined as the linear combination of two nonnegative controls associated with two distinct values of ν_1 in Lemma 6. □

Remark 5. Lemma 6 and Proposition 7 are the only steps in the proof of Theorems 3 and 2 where the nature of the class $\mathcal{U}$ is used. Actually, the proofs of both results are still valid under the following assumptions on the class $\mathcal{U}$: there exists a subclass $\mathcal{V}$ of $\mathcal{U}$ which is convex and closed under concatenation, which contains the null function, and such that Lemma 6 holds with $\cup_{T>0}\mathscr{U}_T = \mathcal{V}$.

4.2 Phase tuning

The controllability of (16), ensured by the Lie–Galerkin Control Condition, together with Proposition 7 implies approximate controllability up to phases of system (2). In this section we show how to correct the dephasing term $e^{\gamma A}$ appearing in Proposition 7 by letting the system evolve freely for a suitable amount τ of time, as proved below (see Figure 1).

Lemma 8. *Let $\psi \in \mathcal{H}$, $\mu > 0$, $N \in \mathbb{N}$, and $\mathcal{N}$ be a neighborhood of $\Pi_N(e^{\mu A}\psi)$ in $\mathrm{span}\{\phi_1, \ldots, \phi_N\}$. Then there exists $\tau \geq 0$ such that $e^{\tau A}\mathcal{N}$ is a neighborhood of $\Pi_N\psi$ in $\mathrm{span}\{\phi_1, \ldots, \phi_N\}$.*

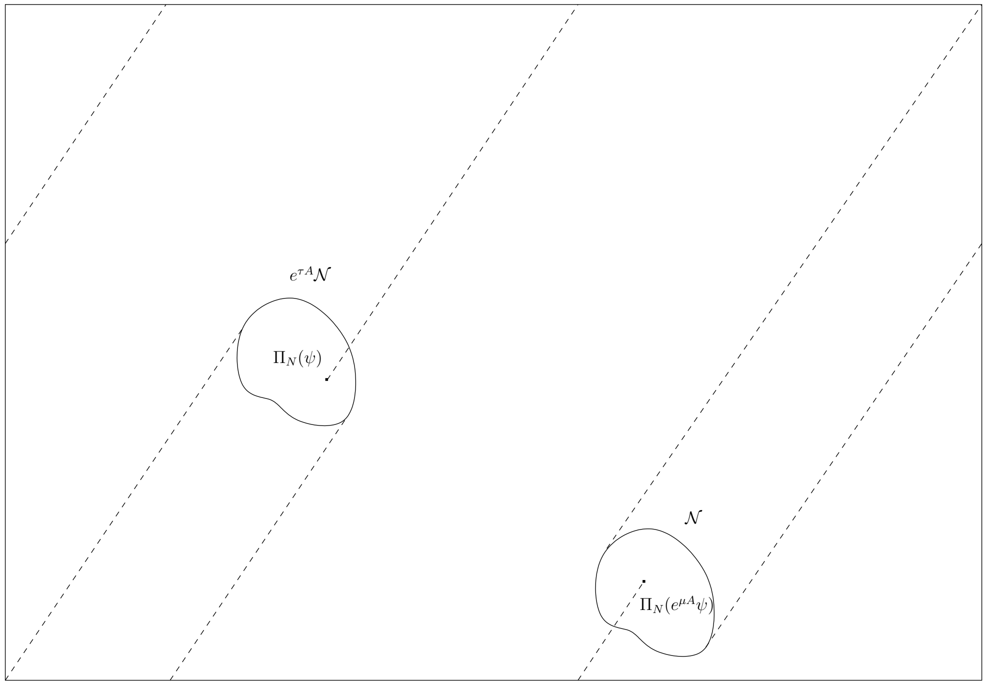

Figure 1: Graphic representation of the statement of Lemma 8. The main idea underlying phase tuning is that the dephasing between solutions of (2) and (5) is along A.

Proof. Without loss of generality we can assume that $\mathcal{N}$ is an open ball $B_{2\varepsilon}(\Pi_N(e^{\mu A}\psi))$ of radius 2ε centered at $\Pi_N(e^{\mu A}\psi)$, for some $\varepsilon > 0$.

Now, consider the sequence $(\beta_k)_{k\in\mathbb{N}}$ of sets

$$\beta_k = B_\varepsilon(\Pi_N(e^{k\mu A}\psi)).$$

All the elements of the sequence $(\beta_k)_{k\in\mathbb{N}}$ are of constant positive volume and are contained in the common subset $B_{\|\Pi_N(\psi)\|+\varepsilon}(0)$ of $\mathrm{span}\{\phi_1,\ldots,\phi_N\}$. Since the latter has finite volume, there exist two integers $\ell, m \geq 1$ such that $\beta_\ell \cap \beta_{\ell+m} \neq \emptyset$. Since $e^{-\ell\mu A}$ is an isomorphism of $\mathrm{span}\{\phi_1,\ldots,\phi_N\}$ we deduce that

$$B_\varepsilon(\Pi_N(\psi)) \cap B_\varepsilon(\Pi_N(e^{m\mu A}\psi)) \neq \emptyset.$$

Therefore

$$\Pi_N(\psi) \in B_{2\varepsilon}(\Pi_N(e^{m\mu A}\psi)) = e^{(m-1)\mu A}B_{2\varepsilon}(\Pi_N(e^{\mu A}\psi)) = e^{(m-1)\mu A}\mathcal{N},$$

which proves the lemma with $\tau = (m-1)\mu$. □

4.3 Final step

Proof of Theorem 2. Let ψ_0 and ψ_1 be in the unit sphere of $\mathcal{H}$. For $\varepsilon > 0$ let $n_0 \in \mathbb{N}$ be such that

$$\left\|\psi_0 - \frac{\Pi_{n_0}\psi_0}{\|\Pi_{n_0}\psi_0\|}\right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \left\|\psi_1 - \frac{\Pi_{n_0}\psi_1}{\|\Pi_{n_0}\psi_1\|}\right\| < \frac{\varepsilon}{3}.$$

The Lie–Galerkin condition ensures the existence of $n > n_0$ such that system (16) is controllable. By Proposition 7 there exist $u \in \mathcal{U}$, $T_u \geq 0$, and $\gamma \geq 0$ such that

$$\left\| \Upsilon^u_{T_u} \left(\frac{\Pi_{n_0}\psi_0}{\|\Pi_{n_0}\psi_0\|} \right) - e^{\gamma A} \frac{\Pi_{n_0}\psi_1}{\|\Pi_{n_0}\psi_1\|} \right\| < \frac{\varepsilon}{3}.$$

By triangular inequality

$$\|\Upsilon^u_{T_u}(\psi_0) - e^{\gamma A}\psi_1\| < \varepsilon.$$

Applying Lemma 8 there exists a time $\tau \geq 0$ such that $e^{\tau A}\Upsilon^u_{T_u}(\psi_0)$ is ε-close to ψ_1. The concatenation of u and the zero function for a time τ then steers ψ_0 into a ε- neighborhood of ψ_1. Notice that such a concatenation is admissible since, in the case in which $\mathcal{U}$ is the set of C^∞ functions with values in U, the control u, given by Proposition 7, is zero with all derivatives at T_u. $\square$

5 Proof of Theorem 3

5.1 Exact controllability in projections implies approximate controllability

Theorem 3 states that under the Lie–Galerkin Control Condition it is possible to control approximately system (2) and, at the same, to control it exactly in projection on a prescribed number of components. The first step in the proof consists in showing that the approximate controllability part is a consequence of the exact controllability in projections.

Lemma 9. *Assume that for every $n_0 \in \mathbb{N}$, $\psi_1 \in \mathcal{H}$ with $\|\psi_1\| = 1$, and $\psi_2 \in \mathcal{H}$ such that $\|\psi_2\| = 1$ and $\|\Pi_{n_0}(\psi_2)\| < 1$, there exist $u \in \mathcal{U}$ and $T > 0$ such that*

$$\Pi_{n_0}(\Upsilon^u_T(\psi_1)) = \Pi_{n_0}(\psi_2). \tag{18}$$

Then for every $\mathfrak{n} \in \mathbb{N}$, $\psi_{\mathrm{in}} \in \mathcal{H}$ with $\|\psi_{\mathrm{in}}\| = 1$, and $\psi_{\mathrm{f}} \in \mathcal{H}$ such that $\|\psi_{\mathrm{f}}\| = 1$ and $\|\Pi_{\mathfrak{n}}(\psi_{\mathrm{f}})\| < 1$, and for every $\varepsilon > 0$, there exist $u \in \mathcal{U}$ and $T > 0$ such that

$$\Pi_{\mathfrak{n}}(\Upsilon^u_T(\psi_{\mathrm{in}})) = \Pi_{\mathfrak{n}}(\psi_{\mathrm{f}}) \quad \text{and} \quad \|\Upsilon^u_T(\psi_{\mathrm{in}}) - \psi_{\mathrm{f}}\| < \varepsilon.$$

Proof. Let $\mathfrak{n}$, ε, ψ_{in}, and ψ_{f} as above. If $\psi_{\mathrm{f}} \in \mathcal{H}$ has an infinite number of non-zero components then it is sufficient to take $n_0 \geq \mathfrak{n}$ such that $\|\Pi_{n_0}\psi_{\mathrm{f}} - \psi_{\mathrm{f}}\| < \varepsilon/2$ and apply (18) with $\psi_1 = \psi_{\mathrm{in}}$ and $\psi_2 = \psi_{\mathrm{f}}$.

If, instead, there exists $n_0 \in \mathbb{N}$ such that $\langle \phi_{n_0}, \psi_{\mathrm{f}} \rangle \neq 0$ and $\langle \phi_k, \psi_{\mathrm{f}} \rangle = 0$ for every $k > n_0$ then note that $n_0 > \mathfrak{n}$ and consider ψ_2 defined component-wise as

$$\langle \phi_k, \psi_2 \rangle = \begin{cases} \langle \phi_k, \psi_{\mathrm{f}} \rangle & \text{if } k = 1, \dots, n_0 - 1, \\ \sqrt{1-\eta^2}\langle \phi_{n_0}, \psi_{\mathrm{f}} \rangle & \text{if } k = n_0, \\ \eta \langle \phi_{n_0}, \psi_{\mathrm{f}} \rangle & \text{if } k = n_0 + 1, \\ 0 & \text{if } k > n_0 + 1, \end{cases}$$

for $\eta < \varepsilon/2$. Then apply (18) with $\psi_1 = \psi_{\mathrm{in}}$. $\square$

5.2 A useful topological tool

The following topological result is standard in degree theory. We provide its proof for completeness.

Lemma 10. *Let $X \subset \mathbb{R}^n$ be open and bounded and let $F \in C(\overline{X}, \mathbb{R}^n)$ be a homeomorphism between X and $F(X)$. Assume that y_0 is in $F(X)$ and consider $0 < \varepsilon \le \operatorname{dist}(y_0, F(\partial X))$. If $G \in C(\overline{X}, \mathbb{R}^n)$ satisfies*

$$\max_{x\in\partial X} |F(x) - G(x)| < \varepsilon$$

then $y_0 \in \operatorname{int}(G(X))$.

Proof. Consider the homotopy $h \in C([0,1] \times \bar{X}, \mathbb{R}^n)$ defined by $h(t,x) = tF(x) + (1-t)G(x)$. By definition of ε we have that $h(t,x) \neq y_0$ for every $t \in [0,1]$ and $x \in \partial X$. In particular the topological degree $\mathrm{d}(h(t,\cdot), X, y_0)$ is well defined for every $t \in [0,1]$. Since $F|_X$ is a homeomorphism and $y_0 \in F(X)$ then

$$\mathrm{d}(F, X, y_0) \neq 0$$

and by homotopy invariance

$$\mathrm{d}(G, X, y_0) \neq 0$$

which implies that $y_0 \in \operatorname{int}(G(X))$. □

5.3 Normal controllability

In this section we prove that, neglecting the phase, system (5) is normally controllable in projections in the sense of [27].

In this section we consider ψ_1 and ψ_2 unit vectors in $\mathcal{H}$ and $n_0 \in \mathbb{N}$ such that $\|\Pi_{n_0}(\psi_2)\| < 1$. Let $n > n_0$ be such that the Lie–Galerkin Control Condition holds. Let now $\tilde{\psi}_2 \in S^{2n-1}$ be such that $\Pi_{n_0}(\psi_2) = \Pi_{n_0}(\tilde{\psi}_2)$. By classical results of normal controllability (see [18] and [27, Theorem 4.3]) there exist $M_1, \dots, M_k \in \mathcal{W}_n$ and $t_1, \dots, t_k > 0$ such that the map

$$E : (s_1, \dots, s_k) \mapsto e^{s_k M_k} \circ \dots \circ e^{s_1 M_1}(\phi_1)$$

has rank $2n-1$ at $(t_1, \dots, t_k)$ and

$$E(t_1, \dots, t_k) = \tilde{\psi}_2.$$

Since $n > n_0$ and $\|\Pi_{n_0}(\tilde{\psi}_2)\| < 1$, there exist $j_1, \dots, j_{2n_0} \in \{1, \dots, k\}$ such that the map

$$F(s_{j_1}, \dots, s_{j_{2n_0}}) \mapsto \Pi_{n_0}\left(E(t_1, \dots, t_{j_1-1}, s_{j_1}, t_{j_1+1}, \dots, t_{j_{2n_0}-1}, s_{j_{2n_0}}, t_{j_{2n_0}+1}, \dots, t_k)\right)$$

has rank $2n_0$ at $(t_{j_1}, \dots, t_{j_{2n_0}})$ and

$$F(t_{j_1}, \dots, t_{j_{2n_0}}) = \Pi_{n_0}(\psi_2) \quad \left(= \Pi_{n_0}(\tilde{\psi}_2)\right).$$

Now let $\rho > 0$ be such that

$$X := B_\rho(t_{j_1}, \dots, t_{j_{2n_0}}) \subset (0, +\infty)^{2n_0}$$

and F is a diffeomorphism between X and $F(X)$. Let

$$\eta = \inf_{(s_1,\dots,s_{2n_0})\in\partial X} \|F(s_1, \dots, s_{2n_0}) - \Pi_{n_0}(\psi_2)\|, \tag{19}$$

and note that $\eta > 0$.

Lemma 11. *There exist $\gamma \geq 0$ and a map associating with every $(s_1, \ldots, s_{2n_0}) \in \bar{X}$ a control $v^s \in \mathcal{U}$ and $T_s > 0$ such that the image of the mapping*

$$\begin{array}{rccc} G: & \bar{X} & \to & \operatorname{span}\{\phi_1, \ldots, \phi_{n_0}\} \\ & (s_1, \ldots, s_{2n_0}) & \mapsto & \Pi_{n_0}\left(\Upsilon^{v^s}_{T_s}(\psi_1)\right) \end{array}$$

contains $\Pi_{n_0}(e^{\gamma A}\psi_2)$ in its interior.

Proof. Let η be as in (19). By Theorem 2 there exists $w \in \mathcal{U}$ steering ψ_1, $\eta/2$-close to ϕ_1.

Applying Proposition 7 with

$$(\tau_1, \ldots, \tau_k) = (t_1, \ldots, t_{j_1-1}, s_{j_1}, t_{j_1+1}, \ldots, t_{j_{2n_0}-1}, s_{j_{2n_0}}, t_{j_{2n_0}+1}, \ldots, t_k),$$

we deduce the existence of a family of controls u^s, depending continuously on $s \in X$ such that

$$\max_{s \in \bar{X}} \|F(s) - e^{-\gamma A}\Pi_{n_0}\left(\Upsilon^{u^s}_{T_{u^s}}\phi_1\right)\| < \eta/2.$$

Define v^s as the concatenation of w and u^s. The continuity of G with respect to s is ensured by Assumption ($\mathbb{A}5$). The conclusion then follows from Lemma 10. □

5.4 Final step

We are now ready to prove Theorem 3.

Proof of Theorem 3. By Lemma 9 it enough to prove that for every $n_0 \in \mathbb{N}$, $\psi_1 \in \mathcal{H}$ with $\|\psi_1\| = 1$, and $\psi_2 \in \mathcal{H}$ such that $\|\psi_2\| = 1$ and $\|\Pi_{n_0}(\psi_2)\| < 1$, there exist $u \in \mathcal{U}$ and $T > 0$ such that

$$\Pi_{n_0}(\Upsilon^u_T(\psi_1)) = \Pi_{n_0}(\psi_2).$$

Lemma 11 guarantees the existence of a neighborhood $\mathcal{N} \subset G(X)$ of $\Pi_{n_0}(e^{\gamma A}\psi_2)$. By Lemma 8 with $\mu = \gamma$, there exists $\tau \geq 0$ and $\zeta \in \mathcal{N}$ such that $e^{\tau}\zeta = \Pi_{n_0}(\psi_2)$. Let $s \in X$ be such that $\Pi_{n_0}\left(\Upsilon^{v^s}_{T_s}(\psi_1)\right) = \zeta$. The concatenation of v^s and the zero function for a time τ then steers ψ_1 to some ψ_3 such that $\Pi_{n_0}\psi_3 = \Pi_{n_0}\psi_2$. Notice that such a concatenation is admissible since, in the case in which $\mathcal{U}$ is the set of C^∞ functions with values in U, the control v^s, given by Lemma 11 (see also Proposition 7), is zero with all derivatives at T_s. □

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