

# Orientation preserving diffeomorphisms and flows of control-affine systems

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**Abstract:** In this paper we show that for every orientation preserving diffeomorphism of  $\mathbb{R}^d$  there exist time-varying feedback controls such that the diffeomorphism is the flow, at a fixed time, of a bracket generating driftless control-affine system. Then we study possible extensions to systems with drift or with more regular controls using inverse function theorems.

Keywords: Controllability, control-affine systems, time-dependent feedback controls, discrete-time dynamics.

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## 1. INTRODUCTION

In this paper we consider a driftless control-affine system

$$\dot{q} = \sum_{i=1}^m u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d. \quad (1)$$

and we study what kind of dynamics we can realize by an appropriate choice of the time-dependent feedback controls  $(u_1(t, q), \dots, u_m(t, q))$ . In particular, we focus on discrete-time dynamics and, in fact, the problem we treat is the following. Given a diffeomorphism  $P$ , find controls such that the flow of system (1) at a fixed time is equal to  $P$ . When studying dynamics of system (1) it is natural to work with time-varying feedback controls. Indeed, if  $u_i$  are continuous feedback controls not depending on time then we cannot expect system (1) to have locally asymptotically stable equilibria as it was observed in Brockett [1983]. Then J.-M. Coron suggested to use periodic time-varying feedback controls for system (1) and proved that asymptotic stability can be successfully achieved by a smooth time-varying feedback (see Coron [1992, 1995] or [Coron, 2007, section 11.2]). Therefore, since similar results hold true also in the discrete-time case (see Jakubczyk and Sontag [1990] for a parallel between discrete-time and continuous-time nonlinear systems), in order to realize discrete-time dynamics, we need to work with time-varying feedback controls.

Here, assuming  $\{f_1, \dots, f_m\}$  to be bracket generating, we prove that every orientation preserving diffeomorphism of  $\mathbb{R}^d$  can be reached by the flow of system (1). Namely, there exist controls  $(u_1(t, q), \dots, u_m(t, q))$  that are  $C^\infty$  with respect to  $q$  and measurable bounded with respect to  $t$  such that the flow at time 1 of system (1) is exactly the given diffeomorphism. It is remarkable that the required hypothesis is related to the controllability of the system. Indeed, roughly speaking, we prove that the controllability of the system with classical controls (i.e. open loop controls depending on time) depending only on  $t$  implies controllability on the group of orientation preserving diffeomorphisms by means of time-varying feedback controls. To

prove this fact we use a result in Agrachev and Caponigro [2009] which states that every diffeomorphism of a compact connected manifold that is isotopic to the identity can be written as a finite composition of exponentials (i.e. flows at a fixed time) of vector fields in a bracket generating family rescaled by suitable smooth functions.

The structure of the paper is the following. In Section 2 we fix the notations and we recall some classical results in geometric control theory useful in the following. Section 3 is devoted to the statement and the proof of the mentioned result for driftless systems. It is natural and useful for applications to study the case with drift. Moreover, to achieve the controllability of a diffeomorphism, it is usually more convenient to use a smaller family of controls, that is, more regular controls. In Section 4 we address to this question. Unfortunately, the result presented states that a diffeomorphism can be realized as flow at a fixed time only approximately, although in a very strong sense. This result has been proved in Agrachev and Caponigro [2010] and makes use of the classical implicit function theorem applied to the jet of the exponential map. The implicit function theorem allows us to prove surjectivity for this map. Moreover as an application of the fixed point theorem one has that small perturbations of this map remain surjective. In the last section we study analytical properties of the exponential map

$$F(a_1, \dots, a_d) = e^{a_1 X_1} \circ \dots \circ e^{a_d X_d} \Big|_U.$$

for a  $0 \in U \subset \mathbb{R}^d$  and  $X_1, \dots, X_d$  linearly independent at 0. Indeed in Caponigro [2010] the local invertibility of this map, using Nash–Moser implicit function theorem (see Hamilton [1982]), has been proved. Then we present a computation showing the form of the differential of a small perturbation of this map. This could be the starting point to the proof of the fact that the generalized implicit function theorem by Zehnder (Zehnder [1976]) applies to the small perturbation of the map. The motivation comes from the underlying idea of the case of Section 4 and the goal is to prove that the flow of system (1) can reach exactly an orientation preserving diffeomorphism using

controls that are trigonometric polynomials with respect to  $t$ .

## 2. PRELIMINARIES

We denote by  $\text{Diff}(M)$  the group of diffeomorphisms of a smooth  $d$ -dimensional manifold  $M$ , by  $\text{Diff}_0(M)$  the connected component of the identity of  $\text{Diff}(M)$ , and by  $\text{Vec } M$  the space of vector fields on  $M$ . If  $B$  is a neighborhood of the origin in  $\mathbb{R}^d$ , we call  $C_0^\infty(B)$  the closed subspace of real smooth functions from  $B$  to  $\mathbb{R}$  that vanish at the origin.

We assume that  $\text{Diff}(M)$ ,  $\text{Diff}_0(M)$ ,  $\text{Vec } M$ , and  $C^\infty(M)$  are endowed with the standard topology of the uniform convergence of the partial derivatives of any order on any compact subset of  $M$ .

Note that  $\text{Diff}_0(\mathbb{R}^d)$  coincides with the set of orientation preserving diffeomorphisms of  $\mathbb{R}^d$ . Indeed a diffeomorphism isotopic to the identity clearly preserves the orientation. Conversely, let  $P$  be an orientation preserving diffeomorphism of  $\mathbb{R}^d$ . Without loss of generality, we can suppose that  $P$  fixes the origin possibly taking the time dependent translation  $P - tP(0)$ ,  $t \in [0, 1]$ . For simplicity rename  $P - P(0)$  by  $P$  and consider the isotopy  $H(t, q) = P(tq)/t$ , for  $t \in (0, 1]$  and  $H(0, q) = \lim_{t \rightarrow 0} P(tq)/t$ . Since  $P$  is orientation preserving then  $H(0, \cdot)$  belongs to the connected component of the identity of the group of linear invertible operators on  $\mathbb{R}^d$ ,  $GL^+(d, \mathbb{R})$ .

Every diffeomorphism  $P \in \text{Diff}(M)$  naturally defines the following transformation of a vector field  $V$ :

$$\text{Ad } PV(p) = P \circ V \circ P^{-1}, \quad q \in M.$$

In fact,  $\text{Ad } P$  is the linear operator on  $\text{Vec } M$  corresponding to the change of coordinates  $P$ . We also define  $\text{ad } V$  as the linear operator on the algebra  $\text{Vec } M$  that satisfies

$$(\text{ad } V)W = [V, W].$$

We assume that every nonautonomous vector field  $V_t$  under consideration satisfies the growth condition  $V_t(q) \leq \phi(t)(1 + |q|)$ , where  $\phi$  is a locally integrable function. Under this assumption every vector field in this paper can be supposed complete without loss of generality.

For every vector field  $V$  the map which associates with any  $q_0 \in M$  the value of the solution, evaluated at a fixed time  $t$ , of  $\dot{q}(t) = V(q(t))$ , with initial condition  $q(0) = q_0$ , is a diffeomorphism from  $M$  into itself, denoted by  $e^{tV} : q_0 \mapsto e^{tV}(q_0)$ , and called the *flow of  $V$  at time  $t$* . If  $V_\tau$  is nonautonomous vector field, then the map which associates with any  $q_0 \in M$  the value of the solution at a fixed time  $t$  of system

$$\begin{cases} \dot{q}(t) = V_t(q(t)) \\ q(t_0) = q_0, \end{cases}$$

is called (*right*) *chronological exponential* of  $V_\tau$  and it is denoted by

$$\overrightarrow{\exp} \int_{t_0}^t V_\tau d\tau : M \rightarrow M. \quad (2)$$

The map

$$V \in \text{Vec } M \mapsto e^V \in \text{Diff}_0(M)$$

that associates with every vector field its flow at time 1 is called *exponential map*. When there is no ambiguity we will call exponential map also a map that associates with a

nonautonomous vector field its chronological exponential at a fixed time, or, given a control system, the map that associates the control with the flow of the system at a fixed time.

Note that the chronological exponential satisfies the differential equation

$$\frac{d}{dt} \overrightarrow{\exp} \int_0^t V_\tau d\tau = \overrightarrow{\exp} \int_0^t V_\tau d\tau \circ V_t.$$

The chronological notation has been first introduced and developed in Agrachev and Gamkrelidze [1978]. Let us recall some results in chronological calculus (see also Agrachev and Sachkov [2004]) that are useful for what follows.

Let  $P^t = \overrightarrow{\exp} \int_0^t V_\tau d\tau$ , the *variation formula*

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) d\tau = \overrightarrow{\exp} \int_0^t (\text{Ad } P^\tau) W_\tau d\tau \circ P^t, \quad (3)$$

gives a description of the flow of the sum of two nonautonomous vector fields  $V_\tau$  and  $W_\tau$ . Namely, the perturbed flow is a composition of the flow  $P^t$  with the flow of the perturbation  $W^t$  twisted by  $P^t$ .

If  $V_t(s)$  is a nonautonomous vector field smoothly depending on a parameter  $s$ , then from (3) easily follows the identity below, useful to compute the differential of the exponential map. Let  $P^t(s) = \overrightarrow{\exp} \int_0^t V_\tau(s) d\tau$  then

$$\frac{\partial}{\partial s} P^t(s) = \int_0^t (\text{Ad } P^\tau(s)) \frac{\partial}{\partial s} V_\tau(s) d\tau \circ P^t(s). \quad (4)$$

Given a family of vector fields  $\mathcal{F} \subset \text{Vec } M$  we define the orbit of the family through a point  $q_0$ :

$$\begin{aligned} \mathcal{O}_{q_0} &= \{q_0 \circ e^{t_1 f_1} \circ \dots \circ e^{t_k f_k} : t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\} \\ &= \{q_0 \circ P : P \in \text{Gr}\mathcal{F}\}, \end{aligned}$$

where  $\text{Gr}\mathcal{F} = \{e^{t_1 f_1} \circ \dots \circ e^{t_k f_k} : t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\}$ . An important property of orbits comes from this classical result due to Sussmann (see Sussmann [1973])

*Theorem 1.* (Orbit Theorem). The orbit of  $\mathcal{F}$  through each point  $q$  is a connected submanifold of  $M$ . Moreover,

$$T_p \mathcal{O}_q = \text{span}\{q \circ \text{Ad } Pf : P \in \text{Gr}\mathcal{F}, f \in \mathcal{F}\}, \quad p \in \mathcal{O}_q.$$

The importance of this result in control theory comes also from the following theorem that gives a sufficient condition for controllability. Indeed, this classical result, although independent and due to Rashevsky and Chow (see Rashevsky [1938] and Chow [1939]), can be seen as a corollary of the Orbit Theorem.

*Theorem 2.* (Chow – Rashevsky). Let  $M$  be a connected manifold and  $\mathcal{F}$  be a bracket generating family of vector fields. The

$$\mathcal{O}_q = M, \quad \text{for any } q \in M.$$

Note that if  $\mathcal{F}$  is a bracket generating family and  $M$  is connected, then  $\text{Gr}\mathcal{F}$  acts transitively on  $M$ . Namely, for every pair of points  $q_0, q_1 \in M$  there exist an element of  $P \in \text{Gr}\mathcal{F}$  such that  $q_0 = P(q_1)$ . In this case we say that the system  $\mathcal{F}$  is *completely controllable*.

*Remark 1.* If  $\mathcal{F} \subset \text{Vec } M$  is bracket generating then  $\mathcal{O}_q = M$  for every  $q \in M$  and by the Orbit Theorem, for every  $q \in M$ , we have

$$T_q M = \text{span}\{q \circ \text{Ad } Pf : P \in \text{Gr}\mathcal{F}, f \in \mathcal{F}\}.$$

If  $X_1, \dots, X_d$  are such that  $\text{span}\{X_1(q), \dots, X_d(q)\} = T_q M$ , then  $X_i = \text{Ad } P^i f_i$  with  $P^i \in \text{Gr}\mathcal{F}$  and  $f_i \in \mathcal{F}$ . If  $a_1, \dots, a_d \in C^\infty(M)$  then

$$e^{a_1 X_1} \circ \dots \circ e^{a_d X_d} = P_1 \circ e^{b_1 f_1} \circ P_1^{-1} \circ \dots \circ P_d \circ e^{b_d f_d} \circ P_d^{-1},$$

where  $b_i = (P^i)^{-1}(a_i)$ ,  $i = 1, \dots, d$ . In other words the image of the map

$$F : (a_1, \dots, a_d) \mapsto e^{a_1 X_1} \circ \dots \circ e^{a_d X_d},$$

belongs to the group  $\{e^{a_1 f_1} \circ \dots \circ e^{a_k f_k} : a_i \in C^\infty(M), f_i \in \mathcal{F}, k \in \mathbb{N}\}$ .

### 3. DRIFTLESS SYSTEMS

A first answer to our problem of realization of diffeomorphisms comes from the following result of Agrachev and Caponigro [2009].

*Theorem 3.* Let  $M$  be a compact connected manifold and let  $\mathcal{F} \subset \text{Vec } M$  be a family of smooth vector fields. If  $\text{Gr}\mathcal{F}$  acts transitively on  $M$  then there exists a neighborhood  $\mathcal{U}$  of the identity in  $\text{Diff}_0(M)$  and a positive integer  $\mu$  such that every  $P \in \mathcal{U}$  can be presented in the form

$$P = e^{a_1 f_1} \circ \dots \circ e^{a_\mu f_\mu}, \quad (5)$$

for some  $f_1, \dots, f_\mu \in \mathcal{F}$  and  $a_1, \dots, a_\mu \in C^\infty(M)$ .

Consider the driftless control-affine system on a compact manifold  $M$

$$\dot{q} = \sum_{i=1}^m u_i(t, q) f_i(q), \quad q \in M.$$

As a consequence of Theorem 5, we can realize as flow at time 1 any diffeomorphism in  $\text{Diff}_0(M)$  by means of time-varying feedback control as the following proposition states.

*Proposition 4.* Let  $M$  be a compact connected manifold and let  $\{f_1, \dots, f_m\}$  be a bracket-generating family of vector fields. For every  $P \in \text{Diff}_0(M)$  there exist  $m$  time-varying feedback controls, piecewise constant with respect to  $t$ , such that

$$P = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(\tau, q) f_i(q) d\tau.$$

**Proof.** Since  $P \in \text{Diff}_0(M)$  then there exist a path  $\{P^t : t \in [0, 1]\} \subset \text{Diff}_0(M)$  such that  $P^1 = P$  and  $P^0 = \text{Id}$ . For every  $N \in \mathbb{N}$ , consider the diffeomorphism  $P^{k/N} \circ (P^{(k-1)/N})^{-1}$  for  $k = 1, \dots, N$ . By Theorem 3 there exists a neighborhood of the identity  $\mathcal{U} \subset \text{Diff}_0(M)$  such that every diffeomorphism in  $\mathcal{U}$  has a representation of the form (5). Then for  $N \in \mathbb{N}$  sufficiently large  $P^{k/N} \circ (P^{(k-1)/N})^{-1} \in \mathcal{U}$  for every  $k = 1, \dots, N$  and, therefore, there exist  $a_1, \dots, a_{\mu N} \in C^\infty(M)$  such that

$$\begin{aligned} P &= P \circ \left( P^{(N-1)/N} \right)^{-1} \circ P^{(N-1)/N} \circ \dots \circ P^1 \\ &= e^{a_1 f_{i_1}} \circ \dots \circ e^{a_{\mu N} f_{i_{\mu N}}} \end{aligned}$$

with  $i_1, \dots, i_{\mu N} \in \{1, \dots, m\}$ . Now consider, for  $j = 1, \dots, m$  the time-varying feedback controls piecewise constant with respect to  $t$  defined on every interval  $\left[ \frac{k-1}{\mu N}, \frac{k}{\mu N} \right)$ ,  $k = 1, \dots, N\mu$  by

$$u_j(t, q) = \begin{cases} a_k(q) & \text{if } f_j = f_{i_k}, \\ 0 & \text{if } f_j \neq f_{i_k}. \end{cases} \quad (6)$$

Then  $\overrightarrow{\exp} \int_0^1 \sum_{j=1}^m u_j(\tau, q) f_j(q) d\tau = e^{a_1 f_{i_1}} \circ \dots \circ e^{a_{\mu N} f_{i_{\mu N}}}$ . This completes the proof.

Consider now a driftless system on the whole space  $\mathbb{R}^d$  instead of a compact manifold. In this case the exact realization of any diffeomorphism in the connected component of the identity holds true but the controls can no more be assumed to be piecewise constant with respect to  $t$ .

*Theorem 5.* Let  $\{f_1, f_2, \dots, f_m\}$  be a bracket-generating family of vector fields on  $\mathbb{R}^d$ . For any  $P \in \text{Diff}_0(\mathbb{R}^d)$  there exist time-varying feedback controls  $u_1(t, q), \dots, u_m(t, q)$  such that

$$P = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(\tau, \cdot) f_i d\tau.$$

**Proof.** Consider a sequence of compacta  $K_n$  such that

$$K_n \subsetneq K_{n+1}, \text{ and } \bigcup_{n=1}^{\infty} K_n = \mathbb{R}^d.$$

Let  $\varrho_n : \mathbb{R}^d \rightarrow [0, 1]$ , for  $n \geq 1$ , be a sequence of smooth functions such that

$$\varrho_n = \begin{cases} 1 & \text{in } K_{n-1} \\ 0 & \text{in } \mathbb{R}^d \setminus \overset{\circ}{K}_n. \end{cases}$$

and such that  $\varrho' \neq 0$  in  $\overset{\circ}{K}_n \setminus K_{n-1}$ . In particular,  $\varrho_n(q) \rightarrow 1$  as  $n \rightarrow \infty$  for every  $q \in \mathbb{R}^d$ . Since  $P \in \text{Diff}_0(\mathbb{R}^d)$  there exists an isotopy  $H(t, \cdot) \in \text{Diff}_0(\mathbb{R}^d)$ ,  $t \in [0, 1]$  such that  $H(1, \cdot) = P$  and  $H(0, \cdot) = \text{Id}$ . Moreover, up to rescaling the time  $t$  it is possible to assume that  $\frac{\partial}{\partial t} H(t, \cdot) \in \text{Diff}_0(\mathbb{R}^d)$  for every  $t \in [0, 1]$ . Then consider the sequence of diffeomorphisms defined by  $P_n(q) = H(\varrho_n(q), q)$  for every  $n \in \mathbb{N}$ .

By Proposition 4 applied to  $P_n|_{K_n}$  there exist controls  $u_i^n(t, \cdot)$  piecewise constant in  $t$ , such that

$$P_n = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i^n(t, \cdot) f_i dt, \quad \text{on } K_n. \quad (7)$$

We can smoothly extend the controls  $u_i^n(t, \cdot)$  in such a way that  $u_i^n(t, q) = 0$  for  $q \in \mathbb{R}^d \setminus \overset{\circ}{K}_n$  and for every  $t \in [0, 1]$ . Hence (7) holds true on the whole space  $\mathbb{R}^d$ . Moreover, we can choose the controls  $u^n = (u_1^n, \dots, u_m^n)$  in such a way that

$$u^n|_{[0,1] \times K_{n-1}} = u^{n+1}|_{[0,1] \times K_{n-1}}.$$

Indeed, by definition  $P_n = P = P_{n+1}$  on  $K_{n-1}$ , hence at every step the control  $u^{n+1}$  adds informations about the representation on the set  $K_{n+1} \setminus K_n$ . By construction there exist time-varying feedback controls  $u_1, \dots, u_m$  such that

$$\int_0^1 \sum_{j=1}^m u_j^n(t, \cdot) f_j dt \rightarrow \int_0^1 \sum_{j=1}^m u_j(t, \cdot) f_j dt,$$

as  $n$  tends to infinity uniformly with all derivatives on compact sets of  $\mathbb{R}^d$ . Then a classical result in control theory (see [Agrachev and Sachkov, 2004, Lemma 8.2]) guarantees the convergence of the chronological exponentials

$$\overrightarrow{\exp} \int_0^1 \sum_{j=1}^m u_j^n(\tau, \cdot) f_j d\tau \rightarrow \overrightarrow{\exp} \int_0^1 \sum_{j=1}^m u_j(\tau, \cdot) f_j d\tau,$$

as  $n$  tends to infinity, in the same topology. On the other hand  $P_n$  tends to  $P$  as  $n$  goes to infinity and the theorem is proved.

*Remark 2.* Note that the proof of the last theorem applies, without modifications, also to systems on a smooth (possibly non compact) manifold.

#### 4. AN APPROXIMATE RESULT

Theorem 5 provides a positive result for the exact realization of diffeomorphisms as flow at time 1 of a driftless control-affine system using time-varying feedback control. It is natural to ask whether it is possible to extend such a result to systems with drift. In this section we present a result of Agrachev and Caponigro [2010] partially answering this problem. The representation can be achieved for systems with drift and with very regular time-varying feedback control but only approximately, although in a very strong sense.

We denote the  $N$ -jet at 0 of  $a \in C^\infty(\mathbb{R}^d)$  as  $J_0^N(a)$ .  $N$ -jets of vector fields on  $\mathbb{R}^d$  and diffeomorphisms are defined similarly (see, for example [Anosov et al., 1997, Section 2.1]).

*Theorem 6.* Let  $\{f_1, f_2, \dots, f_m\}$  be a bracket generating family of vector fields on  $\mathbb{R}^d$ . Consider the control system

$$\dot{q} = f_0(q) + \sum_{i=1}^m u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d, \quad (8)$$

with controls  $u_i$  such that:

- (i)  $u_i$  is polynomial with respect to  $q \in \mathbb{R}^d$ ;
- (ii)  $u_i$  is a trigonometric polynomial with respect to  $t \in [0, 1]$ ;

for every  $i = 1, \dots, m$ .

Fix  $N, k \in \mathbb{N}$ ,  $\varepsilon > 0$ , and  $B$  a ball of  $\mathbb{R}^d$ . For any  $P \in \text{Diff}_0(\mathbb{R}^d)$ , there exist controls  $u_1(t, q), \dots, u_m(t, q)$  such that, if  $\Phi$  is the flow at time 1 of system (8), then

$$J_0^N(\Phi) = J_0^N(P) \quad \text{and} \quad \|\Phi - P\|_{C^k(B)} < \varepsilon.$$

The strategy of the proof follows four main steps. First, it is shown that the set of diffeomorphisms generated by flows of vector fields in a bracket-generating family closed under multiplication by smooth functions is dense in the group of orientation preserving diffeomorphisms. Then the core of the proof lies in the application of the classical implicit function theorem to the map

$$(a_1, \dots, a_d) \mapsto J_0^N(e^{a_1 X_1} \circ \dots \circ e^{a_d X_d}), \quad (9)$$

for  $a_1, \dots, a_d$  in the space of real polynomials of order  $N$  and for  $X_1, \dots, X_d$  vector fields of  $\mathbb{R}^d$  linearly independent at 0. Note that both the source and the target spaces of map (9) are finite dimensional. Remark 1 guarantees the existence of  $m$  time-varying feedback control  $u_1, \dots, u_m$  such that

$$e^{a_1 X_1} \circ \dots \circ e^{a_d X_d} = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(t, q) f_i(q) dt.$$

The last step consists of an application of Brouwer fixed point theorem. Indeed the implicit function theorem implies that the map (9) has a continuous right inverse. By Brouwer fixed point theorem every map sufficiently close to a continuous map with continuous right inverse is locally surjective too. Up to time reparametrizations the drift  $f_0$  can be suppose arbitrary small so that the jet of the exponential of system (1) can be viewed as a small perturbation of map (9). The same argument with small

modifications allows to assume higher regularity for the controls, using Fourier expansions with respect to time and the density of polynomials in the space of smooth functions.

#### 5. THE EXPONENTIAL MAP

Since Theorem 6 holds true for every  $N$ , it is natural to ask whether it is possible to use its strategy to obtain extensions of Theorem 5. In this section we address to this question. The most natural improvement is to get exact controllability using more regular controls. The first step to this aim consists in providing exact controllability in a neighborhood of the identity in  $\text{Diff}_0(M)$  for driftless systems on a compact connected manifold  $M$  with controls that are trigonometric polynomial.

Here we consider the problem locally in  $\mathbb{R}^d$ . The results of this section can be extend to every compact manifold  $M$  thanks to a classical result of Palis and Smale (see [Palis and Smale, 1970, Lemma 3.1]).

Recall that the strategy of the proof of Theorem 6 is based on the classical implicit function theorem. Unfortunately such tool does not apply for the exponential map neither in dimension  $d = 1$  on the circle  $S^1$ . Indeed the exponential of the 0 vector field is the identity of the group of diffeomorphisms and the derivative of the exponential map at the vector field 0 is the identification of  $\text{Vec } S^1$  with the tangent space of the diffeomorphisms at the identity. Therefore, we have a smooth map of a vector space,  $\text{Vec } S^1$ , to a manifold,  $\text{Diff}(S^1)$ , whose derivative at 0 is the identity map from the vector space to the tangent space of the manifold. If inverse function theorem applied then the exponential map would be locally invertible. Nevertheless the exponential map fails to be locally surjective in a neighborhood of the identity as showed by Hamilton (see [Hamilton, 1982, Example 5.5.2]). The implicit function theorem does not apply for the exponential from  $\text{Vec } S^1$  to  $\text{Diff}(S^1)$  essentially because these spaces are not normed spaces. Indeed, the derivative of an operator in Fréchet spaces may be invertible at one point but not at other points arbitrarily nearby, while in Banach spaces this would follow automatically.

It is not possible to look at the exponential as a map between the Banach spaces  $C^k$ . This is due to the so called “loss of derivatives”. Indeed, while the exponential maps  $C^k$  vector fields into  $C^k$  diffeomorphisms, its differential has an unbounded right inverse since the inverse maps the space  $C^k$  into  $C^{k-1}$ , as showed in Caponigro [2010]. Nevertheless local surjectivity of the exponential map

$$F : C^\infty(U)^d \rightarrow \text{Diff}_0(U) \\ (a_1, \dots, a_d) \mapsto e^{a_1 X_1} \circ \dots \circ e^{a_d X_d} \Big|_U. \quad (10)$$

where  $U \subset \mathbb{R}^d$  with  $0 \in U$  and  $X_1, \dots, X_d \in \text{Vec } \mathbb{R}^d$  are linearly independent at 0, has been proved in [Caponigro, 2010, Lemma 2.1]. The key tool used is the Nash–Moser inverse function theorem. This powerful method is based on an iterated scheme which requires invertibility of the differential of the map (10) not only in one point but in an open set.

*Proposition 7.* There exist  $\varrho > 0$  and an open subset  $\mathcal{U} \subset C_0^\infty(B_\varrho)^d$ , such that the map

$$F(a_1, \dots, a_d) = (e^{a_1 X_1} \circ \dots \circ e^{a_d X_d}) \Big|_{B_\varrho}, \quad (11)$$

is an open map from  $\mathcal{U}$  into  $C_0^\infty(B_\varrho)^d$ , where

$$B_\varrho = \{e^{s_1 X_1} \circ \dots \circ e^{s_d X_d}(0) : |s_i| < \varrho, i = 1, \dots, d\}.$$

### 5.1 An alternative proof of Proposition 4

As a consequence of last proposition and Remark 1 for every  $P \in F(\mathcal{U})$  we have the representation

$$\begin{aligned} P &= P^1 \circ e^{b_1 f_{j_1}} \circ (P^1)^{-1} \circ \dots \circ P^d \circ e^{b_d f_{j_d}} \circ (P^d)^{-1} \\ &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^d u_i(t) b_i f_{j_i} + \sum_{i=1}^m v_i(t) f_i dt, \end{aligned}$$

where  $b_i = (P^i)^{-1}(a_i), i = 1, \dots, d$ . Then, Proposition 7 implies local surjectivity of the map

$$F : b \mapsto \overrightarrow{\exp} \int_0^1 \sum_{i=1}^d u_i(t) b_i f_{j_i} + \sum_{i=1}^m v_i(t) f_i dt, \quad (12)$$

where  $b = b_1, \dots, b_d \in \mathcal{U}' = (P^1, \dots, P^d)^{-1}\mathcal{U}$ . In terms of control-affine systems this means that for every  $P \in F(\mathcal{U})$  there exists time-varying feedback controls  $w_1(t, q), \dots, w_m(t, q)$  piecewise constant with respect to  $t \in [0, 1]$ , such that  $P$  is the flow at time 1 of system

$$\dot{q} = \sum_{i=1}^m w_i(t, q) f_i(q).$$

Moreover we know that the dependence on  $q$  of the controls is, in fact, a linear dependence in the functions  $b_i(q)$ .

*Remark 3.* Actually, by Proposition 4, we know that the result holds true for any given diffeomorphism in the connected component of the identity not just in the open subset  $F(\mathcal{U})$ . But Proposition 7 gives us the additional information that the exponential map (12) has also an invertible differential for every  $b \in \mathcal{U}'$ .

### 5.2 Small perturbations of the exponential map

It is natural, as we did in Section 4, to ask whether it is possible to assume the time-varying feedback controls  $w_1(t, q), \dots, w_m(t, q)$  to be trigonometric polynomials. Applying a fixed point argument as in the finite dimensional case of Section 4 is not possible because of the mentioned ‘‘loss of derivatives’’. On the other hand, in Zehnder [1976] the author provides a statement of an implicit function theorem with quadratic remainder term that does not require the differential to be invertible in an open set but just the existence of an ‘‘approximate right inverse’’ (see [Nirenberg, 2001, Chapter 6]). Indeed the Newton iteration scheme used in the Nash–Moser method applies with this weaker hypothesis. A remarkable example of how the method works with this weaker hypothesis is the conjugacy problem by Moser (see Moser [1966]). The aim of this section is to compute explicitly the differential of a particular perturbation of the exponential map (12). This could be the starting point for the application of the Zehnder version of Nash–Moser implicit function theorem and in particular for the proof of the exact controllability of orientation preserving diffeomorphisms by means of time-varying feedback control which are trigonometric polynomials with respect to time.

Let

$$\begin{aligned} F(b) &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^d u_i(t) b_i f_{j_i} + \sum_{i=1}^m v_i(t) f_i dt \\ &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{d+m} u_i(t) b_i f_{j_i} dt, \end{aligned}$$

provided that  $b_{d+1} = \dots = b_{d+m} = 1$ . Consider the truncated Fourier series of  $u_i(t)$ , say  $u_i^n(t)$ . Then  $u_i^n \rightarrow u_i$ , as  $n \rightarrow \infty$ , in  $L^1[0, 1]$ . Let

$$F_n(b) = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{d+m} u_i^n(t) b_i f_{j_i} dt,$$

then, for every  $b \in \mathcal{U}'$ ,

$$F_n(b) \rightarrow F(b), \quad \text{as } n \rightarrow \infty$$

in the  $C^\infty$  topology (see [Agrachev and Sachkov, 2004, Lemma 8.2]).

Let  $r_i^n(t) = u_i(t) - u_i^n(t)$  and call

$$V_t^n = \sum_{i=1}^{d+m} r_i^n(t) b_i f_{j_i}.$$

We have

$$V_t^n(b) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

in  $L^1[0, 1]$  and uniformly with all derivatives in  $q \in B$ . By variation formula (3),

$$\begin{aligned} F_n(b) &= \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{d+k} u_i(t) b_i f_{j_i} - V_t^n dt \\ &= \overrightarrow{\exp} \int_0^1 \text{Ad } F^t(b) V_t^n(b) dt \circ F(b) \\ &= R_n(b) \circ F(b), \end{aligned}$$

where

$$R_n(b) \rightarrow \text{Id}, \quad \text{as } n \rightarrow \infty.$$

in the  $C^\infty$  topology.

Let us compute the differential of  $F_n$  at a point  $b \in \mathcal{U}'$  applied to the  $d$ -uple of smooth functions  $\xi = (\xi_1, \dots, \xi_d)$ . Using (4), we have

$$\begin{aligned} D_b F_n \xi &= \int_0^1 \text{Ad } F_n^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i} dt \circ F_n(b) \\ &= \int_0^1 \text{Ad}(R_n^t(b) \circ F^t(b)) \sum_{i=1}^d u_i(t) \xi_i f_{j_i} dt \circ \\ &\quad \circ R_n(b) \circ F(b) \\ &= \int_0^1 \text{Ad } R_n^t(b) \circ \text{Ad } F^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i} dt \circ \\ &\quad \circ F(b) \circ \text{Ad } F(b)^{-1} R_n(b). \end{aligned}$$

By definition,

$$R_n(b) = \text{Id} + \int_0^1 R_n^t(b) \circ \text{Ad } F^t(b) V_t^n(b) dt, \quad (13)$$

and

$$\text{Ad } R_n^t(b) = \text{Id} + \int_0^t \text{Ad } R_n^\tau(b) \circ \text{ad}(\text{Ad } F^\tau(b) V_\tau^n(b)) d\tau,$$

therefore the differential of  $F_n$  can be written as

$$\begin{aligned}
D_b F_n \xi &= D_b F \xi + \\
&+ \int_0^1 \left( \int_0^t \text{Ad } R_n^\tau(b) \circ \text{ad}(\text{Ad } F^\tau(b) V_\tau^n(b)) d\tau \right) \circ \\
&\circ \text{Ad } F^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i} dt \circ F_n(b) + \\
&+ \int_0^1 \text{Ad } F^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i} dt \circ \\
&\circ \left( \int_0^1 R_n^t(b) \circ \text{Ad } F^t(b) V_t^n(b) dt \right) \circ F(b),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
D_b F_n \xi &= D_b F \xi \circ \text{Ad } F(b)^{-1} R_n(b) + \\
&+ \int_0^1 \left( \int_0^t \text{Ad } R_n^\tau(b) \circ \text{ad}(\text{Ad } F^\tau(b) V_\tau^n(b)) d\tau \right) \circ \\
&\circ \text{Ad } F^t(b) \sum_{i=1}^d u_i(t) \xi_i f_{j_i} dt \circ F_n(b).
\end{aligned}$$

In other words, up to a small perturbation, the differential of  $F_n$  is an invertible linear operator. It remains to study how the perturbation acts on the linear operator in order to determine whether  $D_b F_n$  has an approximate right inverse or not. The candidate to be an approximate right inverse is  $D_b F \xi \circ \text{Ad } F(b)^{-1} R_n(b)$ . It is remarkable the particular dependence of  $D_b F_n$  on  $b$ , indeed  $b$  appears only in  $F^t(b)$  and  $V_t^n(b)$ .

This problem leads to a great number of other related open problems, such as, to mention just two among the closest, extending the result in order to reach every diffeomorphisms in the connected component of the identity and study whether the result holds true also for a control-affine system with a small drift.

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